P.G. Demidov Yaroslavl State University

Ярославский государственный университет им. П.Г. Демидова

Satellite International Conference on NONLINEAR DYNAMICS & INTEGRABILITY and Scientific School "NONLINEAR DAYS"

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NONLINEAR DYNAMICS & INTEGRABILITY (NDI–2022)

Yaroslavl, June 27 – July 1, 2022

Centre of Integrable Systems

ТЕНЗОР

P. G. Demidov Yaroslavl State University
Regional Scientific and Educational Mathematical Center
“Centre of Integrable Systems”
“Tensor” IT company

NONLINEAR DYNAMICS & INTEGRABILITY (NDI–2022)

Abstracts

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им. П. Г. Демидова

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The conference is held within framework of a development programme for the Regional Scientific and Educational Mathematical Center of the Yaroslavl State University with financial support from the Ministry of Science and Higher Education of the Russian Federation (Agreement on provision of subsidy from the federal budget No. 075-02-2022-886). The scientific school is held within the framework of project No. 21-71-30011 of the Russian Science Foundation.

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CAUCHY INVARIANTS AND EXACT SOLUTIONS OF NONLINEAR EQUATIONS OF HYDRODYNAMICS

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The purpose of the report is to draw attention to an important mathematical object in the Lagrangian description of fluid motion - Cauchy invariants. They reflect the fact of conservation of circulation in an ideal fluid along elementary Lagrangian areas located in three orthogonal planes. These three invariants can be viewed as a vector $\vec{S}$ whose components depend only on the Lagrangian coordinates.

The presentation is based on the principle of comparing Cauchy invariants with known exact solutions in Lagrangian variables – Gerstner waves and their generalizations in hydrodynamics and geophysics, plane Ptolemaic flows and their spatial analogs. At the same time, representations for the Cauchy invariants in the β-plane approximation are found.

In a viscous fluid, the expression for the vector $\vec{S}$ already depends on time. The form of the Navier–Stokes equation in terms of $\vec{S}$ is given, and examples of exact solutions for the motions of a viscous fluid, which can be written in the Lagrangian representation, are analyzed. The expressions of "Cauchy invariants" for them are obtained.

EXISTENCE OF DISCRETE TRAVELING WAVES IN A FULLY CONNECTED RELAY SYSTEM OF MACKEY–GLASS TYPE EQUATIONS

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The Mackey – Glass generator is an electric generator whose operation is described by the Mackey – Glass equation

$$\frac{dV}{dt} = -bV + \frac{acV(t-\tau)}{1 + (cV(t-\tau))^{\gamma}} \tag{1}$$

Here, $V(t)$ is the voltage function, $a > 0$ is the saturation level of nonlinearity, $b > 0$ is the RC constant, $\tau > 0$ is the time delay, the parameter $\gamma > 0$ determines the shape of the nonlinear function, and $c > 0$ is the feedback strength.

We consider a model of a fully connected circuit of $m+1$ generators

$$\frac{dV}{dt} = -bV(t) + \frac{ac\left(V(t-\tau) + \sum_{k=1}^{m} V(t-h_k)\right)}{1 + \left(c\left(V(t-\tau) + \sum_{k=1}^{m} V(t-h_k)\right)\right)^{\gamma}}, \tag{2}$$

where $h_k > 0$ are the time delays, and the other parameters of the equation (2) have the same meaning as in the equation (1).

Let $u(t)$ be a function such that $V(t) = c^{-1}u\left(\frac{t}{\tau}\right)$. We renormalize the parameters $b = \frac{b}{\tau}$, $h_k \mapsto \frac{h_k}{\tau}$ and the time $t \mapsto \frac{t}{\tau}$.

Assume γ is a large parameter. As $\gamma \to +\infty$, we obtain the following relay system

$$\dot{u} = -bu + aw(t)F(w(t)), \tag{3}$$

where

$$w(t) = u(t-1) + \sum_{i=1}^{m} u(t-h_i)$$

and

$$F(w) \stackrel{\text{def}}{=} \lim_{\gamma \to +\infty} \frac{1}{1+w^\gamma} \begin{cases} 1, & 0 < w < 1, \\ \frac{1}{2}, & w = 1, \\ 0, & w > 1. \end{cases}$$

In this work, we seek a stable periodic solution of the system (3) in the form of a discrete traveling wave.

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LIBRATIONS AND ASYMPTOTICS OF SPECTRAL BANDS AND GAPS OF TRIGONALLY SYMMETRIC DIMERS

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We discuss exponentially small semi-classical asymptotics (tunneling) of low lying small spectral bands and gaps for a Schrödinger operator describing a dimer in a trigonally symmetric potential field. This problem has four degrees of freedom, and the potential is periodic in two variables, and grows to infinity in the other two. We show how this problem can be reduced to the simplest tunneling problem, namely, the energy splitting in the double well. We discuss the role of classical heteroclinic and libration motions in the obtained asymptotic formulas. Also, we show that these formulas are constructive, i.e. can be implemented in practice with help of variational approach.

This work was funded by the Russian Science Foundation (project No. 21-71-30011).

ON THE GEOMETRY OF TRAJECTORIES OF DYNAMICAL SYSTEMS

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The article considers a class of autonomous fast-slow systems of arbitrary dimension with sufficiently smooth right-hand. We show that under certain conditions the product of the curvatures of the trajectories depends on the velocity of points along these. The obtained dependence is used to prove some geometric properties of trajectories of autonomous dynamical systems.

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BRAID GROUPS AND THE YANG–BAXTER EQUATION

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In this talk I discuss some constructions of solutions to the Yang–Baxter equation on $B \otimes C$ (on $B \times C$, respectively) if given (B, R^B) and (C, R^C) are two linear (set-theoretic) solutions of the Yang–Baxter equation. One of the key observation is the relation of this question with the virtual pure braid group. For instance we show that any solution (X, R) to the Yang–Baxter equation with invertible R defines a representation of the virtual pure braid group VP_n, for any $n \geq 2$, into $Aut(X^{\otimes n})$ for linear solution and into $Sym(X^n)$ for set-theoretic solution.

This is join work with D. V. Talalaev.

ON THE CRITICAL RATE OF THE KELVIN-HELMHOLTZ INSTABILITY REALIZATION

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Nonlinear effects have a significant impact on the dynamics of wave motions in fluids. For example, due to nonlinear effects, the phenomenon of Stokes drift arises — directed motion towards the propagation of a wave with a velocity [1]

$$U_{dr} = \zeta^2 \omega k \exp\left(-2kd\right).$$

Here ζ is the amplitude, and ω is the frequency of wave motion, k is the wave number, and d is the depth at which the drift velocity is considered. If the wave motion propagates along the interface of two fluids, then the drift motion occurs in both.

If one fluid moves relative to another with a constant velocity U_0, then at some critical value U_{cr} the interface is destabilized. This type of instability is called Kelvin–Helmholtz instability. In this case, the frequency of wave motion takes complex values. The real part of the frequency is described by the ratio

$$\sigma = Re\left(\omega\right) = \frac{k\rho_a U_0}{\rho_w + \rho_a} \tag{1}$$

and the image part describes by the ratio

$$r = Im\left(\omega\right) = \frac{\sqrt{k^2\rho_a\rho_w U_0^2 - kg\left(\rho_w^2 - \rho_a^2\right) - k^3\gamma\left(\rho_w + \rho_a\right)}}{\rho_w + \rho_a} \tag{2}$$

Here ρ_w is the density of the lower denser fluid, and ρ_a is the density of the upper one; γ is the surface tension coefficient at the interface of fluids; g is the acceleration of gravity.

It can be shown that drift velocity in the lower fluid U_{drw} and in the upper fluid U_{dra} in case $U_0 > U_{cr}$ describes by the ratio

$$U_{drw} = \zeta^2 \sigma k \exp(2rt) \exp(2kz) \tag{3}$$

$$U_{dra} = U_0 + \zeta^2 (\sigma - kU_0) k \exp(2rt) \exp(-2kz) \tag{4}$$

And at the initial stages of the development of the Kelvin–Helmholtz instability, the velocity of the relative motion of the liquid particles contacting along the interface describes by the ratio

$$U_{rel} = U_0 \left(1 - \zeta^2 k^2\right) \exp(2rt) \tag{5}$$

It is shown that due to nonlinear phenomena, the critical rate of realization of the Kelvin-Helmholtz instability turns out to be less than previously thought. At the initial stages of the Kelvin-Helmholtz instability development, drift additives are directed in opposite directions and exponentially increase over time. This leads to a decrease in the relative velocity of individual liquid particles at the interface. And, consequently, leads to the stabilization of the surface.

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GRADED KORTEWEG–DE VRIES HIERARCHY AND POLYNOMIAL INTEGRABLE SYSTEMS

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Integrable Hamiltonian dynamical systems defined by fractional powers of the Schrödinger operator are well known. The talk is devoted to the development of the theory of such systems based on the methods of the theory of graded differential algebras and the theory of cohomology of graded infinite-dimensional Lie algebras.

Consider a *graded* commutative differential polynomial algebra

$$\mathfrak{A}_0 = (\mathbb{C}[u_0, u_1, \ldots],\ D),$$

where D is the derivation of $\mathbb{C}[u_0, u_1, \ldots]$ such that $D(u_k) = u_{k+1}$, $k = 0, 1, \ldots$ We have $|u_0| = 2$, $|D| = 1$ and $|u_k| = k + 2$. Thus,

$$\mathfrak{A}_0 = \bigoplus_{k \geqslant 0} \mathfrak{A}_{0,k}$$

where $\mathfrak{A}_{0,k}$ is a graded linear space generated by homogeneous polynomials of weight k. For further constructions, it is important that the operator D defines monomorphisms $D\colon \mathfrak{A}_{0,k} \to \mathfrak{A}_{0,k+1}$ for all $k > 1$.

The series

$$A = \sum_{i \leqslant m} a_i D^i \ \text{ where } \ a_m \neq 0,\ a_i \in \mathfrak{A}_{0,|a_m|+m-i}\ ,$$

is called a homogeneous pseudodifferential operator of order $m \in \mathbb{Z}$ and weight $|a_m| + m$. The set of all such series defines a graded non-commutative associative algebra $\mathfrak{A}_0^D$ over $\mathbb{C}$.

We explicitly introduce the homogeneous skew-symmetric bilinear over $\mathbb{C}$ form

$$\sigma(\cdot,\cdot)\colon \mathfrak{A}_0^D \otimes \mathfrak{A}_0^D \to \mathfrak{A}_0$$

and prove that it defines a non-trivial two-dimensional cocycle of the graded infinite-dimensional Lie algebra associated with the associative algebra $\mathfrak{A}_0^D$. For any $A, B \in \mathfrak{A}_0^D$, the remarkable formula

$$\sigma\bigl(D, [A, B]\bigr) = D\bigl(\sigma(A, B)\bigr)$$

underlies the construction of the first integrals of the considered dynamical systems. The stationary g-equations of the Korteweg–de Vries hierarchy are widely known as S.P. Novikov's equations. Each such equation defines a $(g+1)$-dimensional family of polynomial Hamiltonian dynamical systems in $\mathbb{C}^{2g}$. In the talk we will show that using form $\sigma(\cdot,\cdot)$ one can explicitly write homogeneous polynomials that define the Hamiltonians of these systems. We will explicitly write the corresponding Poisson brackets.

All constructions and results carry over directly to the case of the graded non-commutative Korteweg–de Vries hierarchy.

The talk based on the papers [1, 2].

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ASYMPTOTIC REDUCTION OF NONLINEAR SHALLOW WATER EQUATIONS WITH FREE BOUNDARY IN A BASIN WITH A SHALLOW BEACH TO A PROBLEM WITH FIXED BOUNDARY

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We discuss an accurate derivation of the asymptotic reduction of nonlinear shallow water equations with free boundary in a basin with a shallow beach to a problem with fixed boundary in one- and two-dimensional cases. It follows a simple practical algorithm for constructing asymptotic solutions of nonlinear shallow water equations consisting of two steps: 1) construction of solutions of linear equations with fixed boundary (coastline) and 2) representation of solutions of nonlinear equations in parametric form based on solutions of step 1). The proof is based on some considerations related to the Carrier-Greenspan transformation known in the theory of one-dimensional shallow water equations and on the ideas of the uniformization method for partial differential equations with singularities.

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MULTI-MODE CORRELATIONS AS SIGNATURE OF TURBULENCE

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The general problem is what is the most sensitive and sensible statistical characteristics distinguishing turbulence from thermal equilibrium. We consider turbulence of capillary water waves and the so-called Fibonacci chain of interacting modes which models both incompressible turbulence and sets of resonantly interacting waves. We show that in turbulent cascades there exist multi-mode correlations of arbitrary order.

SPHERICAL FLOW DIAGRAM WITH FINITE HYPERBOLIC CHAIN-RECURRENT SET

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In this paper, we consider flows with a finite hyperbolic chain-recurrent set without heteroclinic intersections on arbitrary closed n-manifolds. For such flows, the existence of a dual attractor and a repeller separated by a $(n-1)$-dimensional sphere, which is secant for wandering trajectories in addition to the attractor and repeller, is proved. The study of the dynamics of the considered flows makes it possible to obtain a topological invariant, called a spherical flow scheme, and consisting of a set of multi-dimensional spheres that are the intersection of a secant sphere with invariant saddle manifolds. Note that for some classes the flow spherical scheme is a complete invariant. Thus, it follows from G. Fleitas results that for polar flows (with a single sink and a single source) on the surface, it is the spherical scheme that is a complete equivalence invariant.

The work was supported by the Russian Science Foundation (project No. 21-11-00010), except for section 3.

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ON REAL HAMILTONIAN FORMS OF AFFINE TODA FIELD THEORIES

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We will present real Hamiltonian forms of 2-dimensional Toda field theories related to exceptional simple Lie algebras [1], and the spectral theory of the associated Lax operators. Real Hamiltonian forms [2] are a special type of “reductions” of Hamiltonian systems, similar to real forms of semi-simple Lie algebras. Examples of real Hamiltonian forms of affine Toda field theories related to exceptional complex untwisted affine Kac-Moody algebras will be presented.

Along with the associated Lax representations, we will also discuss the relevant Riemann-Hilbert problems and derive the minimal sets of scattering data that determine uniquely the scattering matrices and the potentials of the Lax operators.

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SOLITON VERSUS THE GAS

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I report on the analysis of infinite soliton solutions and their properties in two models, the modified Korteweg de Vries equation and the nonlinear Schrodinger equation. In the former case I study the interaction of a single larger soliton with the soliton gas and show that the leading order average velocity of the larger soliton satisfies Zakharov-El integral equation.

Based on the works [1, 2].

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DETERMINATION OF THE HOMOTOPY TYPE OF A MORSE-SMALE DIFFEOMORPHISM ON AN ORIENTABLE SURFACE BY A HETEROCLINIC INTERSECTION

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This report is devoted to the study of homotopy types of orientation-preserving Morse-Smale diffeomorphisms on closed orientable surfaces. Since any Morse-Smale diffeomorphism has a finite set of periodic points, then, according to the Nielsen-Thurston classification, it is homotopic to either a periodic homeomorphism or an algebraically finite order homeomorphism. It follows from the results of V. Grines and A. Bezdenezhnykh that any gradient-like diffeomorphism is homotopic to a periodic homeomorphism. However, when the wandering set of a given diffeomorphism contains heteroclinic intersections, then the question of its homotopy type has remained open until now.

In the present work, it is proposed an algorithm for recognizing the homotopy type of a non-gradient-like Morse-Smale diffeomorphism by its heteroclinic intersection. The algorithm is based on the construction of a filtration for a diffeomorphism and calculation of the intersection index of saddle separatrices in the fundamental annuli of filtration elements. It is established that a Morse-Smale diffeomorphism is homotopic to a periodic homeomorphism if and only if the total intersection index over all homotopic annuli is equal to zero.

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THE ANALYTICAL MODEL FOR THE CONTACT FORCE DISTRIBUTION IN DISORDERED PACKINGS OF PARTICLES

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The packing of hard-core particles in contact with their neighbors is considered as the simplest model of disordered particulate media. We formulate the statically determinate problem which allows analytical investigation of the statistical distribution of contact force magnitude. The toy-model of the Boltzmann type equation for the probability of contact force distribution is formulated and studied. An experimentally observed exponential distribution is derived.

The work was supported by the Russian Science Foundation (grant No. $21-71-30011$) and was carried out within the framework of a development programme for the Regional Scientific and Educational Mathematical Center of the Yaroslavl State University with financial support from the Ministry of Science and Higher Education of the Russian Federation (Agreement on provision of subsidy from the federal budget No. $075-02-2022-886$).

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APPROXIMATE FINITE-GAP FORMULAS FOR THE SPATIALLY DOUBLE-PERIODIC DAVEY-STEWARDSON-2 ROGUE WAVES

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The study of anomalous waves (rogue waves) in non-linear systems is one of important areas of modern mathematical physics. In particular, the soliton equations are actively used as models for generation of rogue waves. The is a huge literature, dedicated to the applications of 1+1-dimensional soliton systems to the anomalous waves theory, but the 2+1 dimensional case is studied much less.

One of the main 2+1 soliton systems, admitting rogue waves type solutions is the focusing Davey-Stewardson 2 equation.

In a recent series of papers we constructed asymptotic formulas for spatially-periodic focusing Nonlinear Schrodinger equation under assumption that the Cauchy data is a small perturbation of an unstable background. Such solutions are used as models for generation of rogue waved in non-linear systems. Due to the fact that the spectral curves are close to rational ones, all ingredients of the theta-functional formulas can be explicitly calculated in the leading order. These asymptotic solutions are expressed in terms of elementary functions and provide a good approximation for the exact ones. In the present work we construct analogous formulas for the focusing Davey-Stewardson 2 equation.

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ON THE INTEGRABLE CLASSIFICATION OF 3D LATTICES

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In our recent studies (see, for instance, [1]-[4]) we found that all known Toda-type integrable lattices

$$u^j_{x,y} = f(u^j_x, u^j_y, u^{j+1}, u^j, u^{j-1}), \tag{1}$$

integrable semidiscrete models with two discrete and one continuous independent variables, note that they all have the form

$$u^j_{n+1,x} = f(u^j_{n,x}, u^{j+1}_n, u^j_n, u^j_{n+1}, u^{j-1}_{n+1}) \tag{2}$$

as well as integrable fully discrete equations, all known examples can be represented as

$$u^j_{n+1,m+1} = f(u^{j+1}_{n+1,m}, u^j_{n+1,m}, u^j_{n,m}, u^j_{n,m+1}, u^{j-1}_{n,m+1}) \tag{3}$$

have the following property. They can be reduced to systems of hyperbolic type differential, differential-difference or, respectively, discrete equations by imposing at the points $j = 0$ and $j = N$ for an arbitrary $N > 0$ cutoff boundary conditions of a special kind, so that the resulting systems are integrable in the sense of Darboux. An algorithm for classifying equations of the form (1)-(3) is proposed, taking as a classification feature the presence of a Darboux-integrable reduction hierarchy for the equation. Recall that the Darboux integrability property has a convenient algebraic formulation. A system of hyperbolic type is Darboux integrable if and only if its characteristic Lie-Rinehart algebras corresponding to the characteristic directions are of finite dimension.

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CONSTRUCTION OF EXACT SOLUTIONS TO THE RUIJSENAARS–TODA LATTICE VIA GENERALIZED INVARIANT MANIFOLDS

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In the talk will discuss a new method for constructing algebro-geometric solutions of nonlinear integrable lattices, based on the concept of a generalized invariant manifold (GIM) (see, for instance, [1]–[3]). In contrast to the finite-gap integration method, instead of the eigenfunctions of the Lax operators, we use a joint solution of the linearized equation and GIM. This makes it possible to derive Dubrovin type equations not only in the time variable t, but also in the spatial discrete variable n. We illustrate the efficiency of the method using the Ruijsenaars–Toda lattice as an example, for which we have derived a real and bounded particular solution in the form of a kink, a periodic solution expressed in terms of Jacobi elliptic functions and a solution expressed through Weierstrass $\wp$-function [4].

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CONTINUED FRACTIONS AND HYPERELLIPTIC CURVES

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Following van der Poorten, we consider a family of nonlinear maps that are generated from the continued fraction expansion of a function on a hyperelliptic curve of genus g. Using the connection with the classical theory of J -fractions and orthogonal polynomials, we show that in the simplest case $g = 1$ this provides a straightforward derivation of Hankel determinant formulae for the terms of a general Somos-4 sequence, which were found in a particular form by Chang, Hu, and Xin. We extend these formulae to the higher genus case, and prove that generic Hankel determinants in genus 2 satisfy a Somos-8 relation. Moreover, for all g we show that the iteration for the continued fraction expansion is equivalent to a discrete Lax pair with a natural Poisson structure, and the associated nonlinear map is a discrete integrable system. The connection with the Toda lattice, and also the link to S-fractions via contraction, and a family of maps associated with the Volterra lattice, will briefly be mentioned.

Some of this is joint work with John Roberts and Pol Vanhaecke.

KEPLER DYNAMICS ON A CONFORMABLE POISSON MANIFOLD

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The problem of Kepler dynamics on a conformable Poisson manifold is discussed. The Hamiltonian function is defined and the related Hamiltonian vector field governing the dynamics is derived, leading to a modified Newton second law. Conformable momentum and Laplace-Runge-Lenz vectors are considered, generating $SO(3), SO(4)$, and $SO(1,3)$ dynamical symmetry groups. The corresponding first Casimir of $SO(4)$ is obtained. Then, the hydrogen atom is described in a conformable Hilbert space of complex square integrable functions. The recursion operators are constructed and used to compute the integrals of motion in action-angle coordinates. Main relevant properties are deducted and analyzed.

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SOME ALGEBRAIC AND DIFFERENTIAL-GEOMETRIC METHODS IN INTEGRABLE SYSTEMS WITH APPLICATIONS TO BÄCKLUND TRANSFORMATIONS AND ZAMOLODCHIKOV TETRAHEDRON MAPS

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We present a review of recent results on some algebraic and differential-geometric methods in the theory of integrable systems with applications to Bäcklund transformations for partial differential and difference equations, as well as to Zamolodchikov tetrahedron maps. The main tools in the methods are zero-curvature representations (Lax representations), gauge transformations, jet spaces (jet bundles), Lie algebras (including infinite-dimensional ones), Lie groups, and their actions on manifolds.

Bäcklund transformations (BTs) are well known to be a powerful tool for constructing exact solutions for integrable nonlinear partial differential and difference equations, including soliton solutions. Zamolodchikov tetrahedron maps are set-theoretical solutions to the Zamolodchikov tetrahedron equation, which belongs to fundamental equations of mathematical physics and has applications in many diverse branches of physics and mathematics, including statistical mechanics, quantum field theories, combinatorics, low-dimensional topology, and the theory of integrable systems. Presently, the relations of Zamolodchikov tetrahedron maps with integrable systems and with algebraic structures (including groups and rings) are a very active area of research (see, e.g., [2, 11, 12, 14] and references therein).

The main topics of the talk are the following:

1. Algebraic necessary conditions for existence of a Bäcklund transformation (BT) between two given (1+1)-dimensional evolution partial differential equations (PDEs). Here we consider the most general class of BTs, which are not necessarily of Miura type. The obtained necessary conditions allow us to prove non-existence of BTs between two given equations in many cases.

To obtain these conditions, for (1+1)-dimensional evolution PDEs we find a normal form for zero-curvature representations (ZCRs) with respect to the action of the group of local gauge transformations and define, for a given (1+1)-dimensional evolution PDE $\mathcal{E}$, a family of Lie algebras $F(\mathcal{E})$, whose representations classify all ZCRs of the equation $\mathcal{E}$ up to local gauge transformations. Furthermore, these Lie algebras allow us to prove non-existence of any nontrivial ZCRs for some classes of PDEs.

In our approach, ZCRs may depend on partial derivatives of arbitrary order, which may be higher than the order of the PDE. The algebras $F(\mathcal{E})$ are defined in terms of generators and relations and generalize Wahlquist–Estabrook prolongation Lie algebras, which are responsible for a much smaller class of ZCRs. The structure of the Lie algebras $F(\mathcal{E})$ has been studied for some classes of (1+1)-dimensional evolution PDEs of orders 2, 3, 5, which include Korteweg–de Vries (KdV), modified KdV, Krichever–Novikov, Kaup–Kupershmidt, Sawada–Kotera, nonlinear Schrödinger, and (multicomponent) Landau–Lifshitz type equations. Among the obtained algebras one finds infinite-dimensional subalgebras of Kac–Moody algebras and infinite-dimensional Lie algebras of certain matrix-valued functions on some algebraic curves.

2. A method to construct BTs of Miura type (differential substitutions) for (1+1)-dimensional evolution PDEs, using zero-curvature representations and actions of Wahlquist–Estabrook prolongation Lie algebras. Our method is a generalization of a result of V.G. Drinfeld and V.V. Sokolov [3] on BTs of Miura type for the KdV equation.

3. A method to construct BTs of Miura type for differential-difference (lattice) equations, using Lie group actions associated with Darboux–Lax representations of such equations. The considered examples include Volterra, Narita–Itoh–Bogoyavlensky, Toda, and Adler–Postnikov lattices. Applying our method to these examples, we obtain new integrable nonlinear differential-difference equations connected with these lattices by BTs of Miura type.

4. As said above, (Zamolodchikov) tetrahedron maps are set-theoretical solutions to the Zamolodchikov tetrahedron equation. We prove that the differential of a tetrahedron map on a manifold is a tetrahedron map as well. Using this result and Lax representations, we develop methods to obtain new tetrahedron maps from known ones. Also, we present new tetrahedron maps on arbitrary groups and on associative rings. Liouville integrability is established for some of the constructed maps.

Some results of the talk are based on joint works with G. Manno [8–10], with G. Berkeley [1], with V. Kolesov, S. Konstantinou-Rizos, M. M. Preobrazhenskaia [6, 7], as well as on the papers [4, 5]. In the study of Bäcklund transformations for PDEs we use the theory of coverings of PDEs developed by A. M. Vinogradov and I. S. Krasilshchik [13, 15].

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ASYMPTOTICS OF SOLUTIONS OF ONE NONLINEAR EQUATION WITH DELAY

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In this study we consider a nonlinear equation

$$\dot{u} + u = \lambda F(u(t-T)), \tag{1}$$

where u is a scalar function, $T > 0$, $\lambda \gg 1$,

$$F(u) = \begin{cases} b, \quad u \le p_L, \\ f(u), \quad p_L < u < p_R, \\ d, \quad u \ge p_R, \end{cases}$$

values p_L and p_R satisfy inequality $p_L < 0 < p_R$, b and d are constants ($bd \neq 0$).

We study asymptotics of all solutions of equation (1) with initial conditions from two sets: S_+ and S_-. The set S_+ (S_-) consists of continuous functions $u(s)$ ($s \in [-T, 0]$) such that $u(s) > p_R$ ($u(s) < p_L$) for all $s \in [-T, 0)$ and $u(0) = p_R$ ($u(0) = p_L$ respectively).

We construct asymptotics of all solutions and find conditions on signs of parameters b and d for existence of relaxation cycle. We find amplitude and period of these cycles.

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CONSTRUCTION OF QUASINORMAL FORMS IN THE PROBLEM OF VIBRATIONS OF PEDESTRIAN BRIDGES

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In [1], in connection with the study of the stability of pedestrian suspension bridges, a model was proposed that takes into account the influence of pedestrians on vibrations of constructions

$$
\begin{aligned}
&\ddot{u}_j + \lambda(\dot{u}_j^2 + u_j^2 - \varepsilon)\dot{u}_j + \omega^2 u_j = -\ddot{y}_j,\\
&\ddot{y}_j + 2h\dot{y}_j + \Omega^2 y_j = -\frac{r}{N}\sum_{i=1}^{N}\ddot{u}_i,
\end{aligned}
\tag{1}
$$

where $j = 1, \ldots, N$. All parameters of this walker-bridge model are positive. They are described in [1].

Several analytical results on the collective behavior of a chain of coupled oscillators will be presented. (1).

By $u_j(t),\ y_j(t)$ we denote the functions of two variables $u(t, x_j)$ and $y(t, x_j)$ respectively. Here, $x_j \in [0, 1]$ are points uniformly distributed on some neighborhood with angular coordinate $x_j = 2\pi N^{-1} j$.

There are two main assumptions that open the way to the employment of analytical methods. First, we assume that the number of oscillators (pedestrians) in, i. e. $N \gg 1$. This gives grounds to move from a discrete system with respect to $u(t, x_j),\ y(t, x_j)$ to a continuous spatially distributed boundary value problem for the quantities $u(t, x)$, $y(t, x)$

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} + \lambda\left(u^2 + \left(\frac{\partial u}{\partial t}\right)^2 - \varepsilon\right)\frac{\partial u}{\partial t} + \omega^2 u = -\frac{\partial^2 y}{\partial t^2},\\
&\frac{\partial^2 y}{\partial t^2} + 2h\frac{\partial y}{\partial t} + \Omega^2 y = -r\int_0^1 \frac{\partial^2 u(t,s)}{\partial t^2}\,ds,
\end{aligned}
\tag{2}
$$

$$
u(t, x+1) \equiv u(t, x), \quad y(t, x+1) \equiv y(t, x). \tag{3}
$$

The second restriction is that the ε parameter is small enough:

$$0 < \varepsilon \ll 1. \tag{4}$$

Note that, under this condition, the van der Pol equation

$$\ddot{u} + \lambda[\dot{u}^2 + u^2 - \varepsilon]\dot{u} + \omega^2 u = 0$$

has a stable cycle $u_0(t, \varepsilon) = \varepsilon^{1/2}\rho_0 \cos(\omega t(1 + O(\varepsilon))) + O(\varepsilon^{3/2})$ with period $2\pi(\omega + O(\varepsilon))^{-1}$, where $\rho_0 = (3\omega^2 + 1)^{-1/2}$.

Under the condition (4) we study the behavior of all solutions of the boundary value problem (2), (3) with initial conditions from some sufficiently small neighborhood of the zero equilibrium state independent of ε. Critical cases in the study of the stability of the equilibrium state in this system have infinite dimension.

As the main result, special quasinormal forms will be constructed whose nonlocal dynamics determines the local behavior of solutions to the original boundary value problem. For example, under the condition that the parameter r is small, i.e.

$$r = \varepsilon r_1 \tag{5}$$

the quasinormal form has the form

$$\begin{gathered}\frac{\partial \xi}{\partial \tau} = \frac{1}{2}\lambda\xi + \gamma \int\limits_0^1 \xi(\tau, s)ds + b\xi|\xi|^2, \\ \xi(\tau, x + 1) \equiv \xi(\tau, x).\end{gathered} \tag{6}$$

For coefficients γ and b we have the expression

$$\gamma = r_1\omega^2\Big[2(\Omega^2 - \omega^2 + 2i\omega h)\Big]^{-1}, \quad b = -\frac{1}{2}\lambda(3\omega^2 + 1).$$

The following statement says that the boundary value problem (6) is a quasinormal form.

Theorem 1 *Let the condition* (5) *be satisfied and the boundary value problem* (6) *has a solution* $\xi(\tau, x)$*. Then, the expression*

$$u(t, x) = \varepsilon^{1/2}\Big(\xi(\tau, x)\exp(i\omega t) + \bar{\xi}(\tau, x)\exp(-i\omega t)\Big), \ y(t, x) = 0 \tag{7}$$

satisfies the original boundary value problem (2), (3) *up to* $O(\varepsilon^{3/2})$.

It is possible to find explicitly some important classes of solutions to the boundary value problem (6). For example, families of solutions that are periodic in τ and 2π-periodic and piecewise continuous in the space variable can be defined.

If the parameter r is not small, then the quasinormal form in the case close to the critical one has the form

$$2\frac{d\xi}{d\tau} = \lambda\xi - \lambda(1+3\omega^2)(\xi|\xi|^2 - M(\xi|\xi|^2)) \tag{8}$$

with conditions

$$\xi(\tau, x+1) \equiv \xi(\tau, x), \quad M(\xi(\tau, x)) = 0, \tag{9}$$

where

$$M(v(x)) = \int_0^1 v(x)dx.$$

Here, for the quasinormal form (8), (9), there is an analog of Theorem 1.

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HIERARCHIES OF COMPATIBLE MAPS AND INTEGRABLE NON-ABELIAN DIFFERENCE SYSTEMS

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We introduce families of non-Abelian compatible maps associated with Nth order discrete spectral problems. In that respect we have hierarchies of families of compatible maps that in turn are associated with hierarchies of set-theoretical solutions of the 2-simplex equation a.k.a Yang-Baxter maps. These hierarchies are naturally associated with integrable difference systems with variables defined on edges of an elementary cell of the $\mathbb{Z}^2$ graph, that in turn lead to, Bäcklund related hierarchies of difference systems with variables defined on vertices of the same cell.

ELECTRICAL NETWORKS AND THE POSITIVE PART OF LAGRANGIAN GRASSMANNIAN

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A planar circular graph Γ together with a weight function $\omega : E(\Gamma) \to \mathbb{R}_{\geq}$ is called a *elctrical network* $e(\Gamma, \omega)$ if the following holds:

- all vertices are divided into the set of interior vertexes and the set of exterior or boundary vertices and all boundary vertices lie on a circle;
- boundary vertices are enumerated counterclockwise;
- every edge of Γ is equipped with a positive weight which is the conductivity of this edge .

When electrical voltage is applied to the boundary vertices it extends to the interior vertexes and the electrical current will run through the edges of Γ. These voltages and electrical currents are defined by the Kirchhoff laws [1].

Despite that theory of electrical networks is well-developed it recently has become the source for new significant results related to the cluster algebras, integrable dynamical systems and etc. Particularly noteworthy Lam's approach [2] allowed to link theory of electrical networks with theory of the positive Grassmannians, which theory arised from the fundamental Postnikov's work [3].

In the focus of my speech will be the natural continuation of Lam's results. Using electrical response matrices I will construct embedding of the set of electrical networks to the positive part of Lagrangian Grassmannian and discuss techniques related with this embedding: groves; Kenyon-Wilson theorem about ratios of the special grove partition functions; adding a boundary spike or a bridge to an electrical network and etc.

The report will be based on the joint article [4].

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NONCOMPACT FOLIATIONS OF SEVERAL INTEGRABLE SYSTEMS IN PSEUDO-EUCLIDEAN SPACE

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In recent years, a wide class of integrable systems with noncompact singularities and noncomplete Hamiltonian flows was studied using topological and geometrical approaches. Constructed by A.Fomenko, his scientific school and collaborators [1], the topological classification theory of integrable systems describes mainly the compact case and has been successfully applied to various problems from mechanics, mathematical physics and geometry.

Development of its analog, i.e. a general theory for a natural (in some sense) class of integrable systems with noncompact foliations and noncomplete flows, seems to be a highly non-trivial problem. Several theoretical results on foliations of such systems have been already obtained, e.g., by E.Kudryavtseva (an analog of the Liouville theorem), K.Aleshkin (foliations of noncomplete flows) and S.Nikolaenko (singularities of noncompact integrable systems with 1 and 2 d.o.f.). A brief review on singularities appeared in such systems and their foliations can be found in [2].

Noncompactness property of Liouville fibers was discovered for a number of concrete systems from geometry, mechanics and mathematical physics. Among them, an analog of the Goryachev integrable case was studied by S.Nikolaenko.

In a recent paper by A.Borisov and I.Mamaev [3], a class of Pseudo-Euclidean analogs of integrable systems from rigid body dynamics and related area was considered. The talk is devoted to the author's results on topological properties of these systems. As it turns out, their Liouville foliations with both compact and noncompact fibers contains also noncompact noncritical singularities. Obtained results on the Euler case and new results on Kovalevskaya and Lagrange case analogs will be discussed.

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A DISCRETE DARBOUX–LAX SCHEME FOR INTEGRABLE DIFFERENCE EQUATIONS

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We propose a discrete Darboux–Lax scheme for deriving auto-Bäcklund transformations and constructing solutions to quad-graph equations that do not necessarily possess the 3D consistency property. As an illustrative example we use the Adler-Yamilov type system which is related to the nonlinear Schrödinger equation. In particular, we construct an auto-Bäcklund transformation for this discrete system, its superposition principle, and we employ them in the construction of the one- and two-soliton solutions of the Adler–Yamilov system.

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STEP SOLUTIONS OF QUASI-NORMAL FORM FOR A BOUNDARY VALUE PROBLEM WITH MEAN VALUE

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Consider the system of equations

$$\frac{du}{dt} = (A_0 + \varepsilon A_1)u + F_2(u,u) + F_3(u,u,u) + \varepsilon D_0 \frac{1}{2\pi} \int\limits_0^{2\pi} u(t,x)dx,$$

where $u = u(t,x) \in \mathbb{R}^n$, A_0, A_1, D_0 are $n \times n$ matrices, $F_2(*,*)$, $F_3(*,*,*)$ are linear functions of their arguments, ε is small positive parameter.

This system is considered with periodic boundary condition

$$u(t,x+2\pi) \equiv u(t,x).$$

If matrix A_0 has the eigenvalues with negative real parts, then the zero solution of this boundary value problem is asymptotic stable. If matrix A_0 has one zero eigenvalue, then exist an integrable manifold, in which dynamics of this boundary value problem describes by one-dimensional boundary value problem, which calls as quasi-normal form. In some cases this quasi-normal form has the asymptotic stable piecewise smooth solutions in the form of step functions.

In this work we consider the case, when matrix A_0 has the pair of imaginary eigenvalues. For this case we will find the quasi-normal form and consider the dynamics of this quasi-normal form.

Let $A_0 a = i\omega a$, $A_0^* b = -i\omega b$, then the solution of boundary value problem has the form of asymptotic series

$$\begin{aligned} u(t,\tau,\varepsilon) =& \varepsilon^{1/2}(a\xi(\tau,x)\exp(i\omega t) + \overline{a}\overline{\xi}(\tau,x)\exp(-i\omega t)) + \\ &+ \varepsilon(u_{20}|\xi(\tau,x)|^2 + u_{21}\xi^2(\tau,x)\exp(2i\omega t) + \\ &+ \overline{u}_{21}\overline{\xi}^2(\tau,x)\exp(-2i\omega t)) + \ldots, \quad \tau = \varepsilon t. \end{aligned}$$

Substitute it into the system, we obtain the quasi-normal form

$$\frac{\partial \xi}{\partial \tau} = \lambda \xi + \sigma \xi |\xi|^2 + \gamma M(\xi),\ \xi(\tau, x + 2\pi) \equiv \xi(\tau, x).$$

where $\lambda = (A_1 a, b)$, $\sigma = (F_3(a, a, \overline{a}) + F_3(a, \overline{a}, a) + F_3(a, a, \overline{a}) + F_2(u_{20}, a) + F_2(a, u_{20}) + F_2(u_{21}, \overline{a}) + F_2(\overline{a}, u_{21}), b)$, $\gamma = (D_0 a, b)$.

In this work we consider the case where $\lambda = 1 + i\lambda_0$, $\sigma = -1 + i\sigma_0$, γ is real.

Theorem 1. *Quasi-normal form has solutions*

$$\xi(\tau, x) = \pm\sqrt{1 + \gamma}\exp((\lambda_0 + \sigma_0(1 + \gamma))i\tau).$$

These solutions are asymptotic stable, if $\gamma > 0$.

Consider the solution of quasi-normal form

$$\xi(t, x) = \rho(x)e^{i\omega\tau},$$

where

$$\rho(x) = \begin{cases} \rho_1 e^{i\varphi_1}, & x \in [0, \alpha), \\ \rho_2 e^{i\varphi_2}, & x \in [\alpha, 2\pi). \end{cases}$$

By change of variables we can obtain that $\varphi_1 = 0$, $\varphi_2 = \varphi$.

Substitute this solution into quasi-normal form, then we obtain the system of equations

$$\begin{gathered} \rho_1 - \rho_1^3 = -\frac{\gamma}{2\pi}[\alpha\rho_1 + (2\pi - \alpha)\rho_2 \cos\varphi], \\ -\omega\rho_1 + \lambda_0\rho_1 + \sigma_0\rho_1^3 = -\frac{\gamma}{2\pi}(2\pi - \alpha)\rho_2 \sin\varphi, \\ \rho_2 - \rho_2^3 = -\frac{\gamma}{2\pi}[\alpha\rho_1 \cos\varphi + (2\pi - \alpha)\rho_2], \\ -\omega\rho_2 + \lambda_0\rho_2 + \sigma_0\rho_2^3 = \frac{\gamma}{2\pi}\alpha\rho_1 \sin\varphi. \end{gathered}$$

If there is a solution of this system, then exist the piecewise smooth solution of quasi-normal form.

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INSTABILITY OF EQUILIBRIA IN A SOLENOIDAL FORCE FIELD

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A conjecture on the instability of an isolated equilibrium of a mechanical system in a solenoidal force field is considered. We prove several results supporting this conjecture. The results are proved under some generic additional assumptions which guarantee that the equilibrium is isolated. Moreover, we prove the existence of trajectories asymptotically leaving the equilibrium.

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TYPICAL INTEGRABLE HAMILTONIAN SYSTEMS ON A SIX-DIMENSIONAL SYMPLECTIC MANIFOLD

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Fix $s \in \mathbb{N}$ and consider the action of the group $G = \mathbb{Z}_s$ on the plane $\mathbb{R}^2$ whose generator acts by the rotation $z \mapsto e^{2\pi i/s}z$, where $z = x+iy \in \mathbb{C} \approx \mathbb{R}^2$. Consider the Morse functions $F_{s,0} = F^{\pm,\pm}_{s,0}(x,y) = \pm|z|^2 = \pm(x^2+y^2)$ for $s \geq 1$, and $F_{s,0} = F^{+,-}_{s,0}(x,y) = x^2 - y^2$ for $s = 1, 2$ and two families of $\mathbb{Z}_s$-invariant functions $F_{s,k}$, $k = 1, 2$:

$$F_{s,1} = F_{s,1}(x,y,a,\lambda) = \begin{cases} \pm x^2 + y^3 + \lambda y, & s = 1, \\ \pm x^2 \pm y^4 + \lambda y^2, & s = 2, \\ Re(z^3) + \lambda |z|^2, & s = 3, \\ Re(z^s) \pm a_1|z|^4 + \lambda|z|^2, & s \geq 4, \end{cases}$$

$$F_{s,2} = F_{s,2}(x,y,a,\lambda) = \begin{cases} \pm x^2 \pm y^4 - \lambda_2 y^2 + \lambda_1 y, & s = 1, \\ \pm x^2 \pm y^6 + \lambda_2 y^4 + \lambda_1 y^2, & s = 2, \\ Re(z^4) \pm (1+\lambda_2)|z|^4 \pm a_1|z|^6 + \lambda_1|z|^2, & s = 4, \\ Re(z^5) \pm a_1|z|^6 + \lambda_2|z|^4 + \lambda_1|z|^2, & s = 5, \\ Re(z^6) + a_1|z|^6 \pm a_2|z|^8 + \lambda_2|z|^4 + \lambda_1|z|^2, & s \geq 6. \end{cases}$$

Here $\lambda = (\lambda_i) \in \mathbb{R}^k$ is a small parameter, $a = (a_i) \in \mathbb{R}^m$ are *moduli*, $m \in \{0,1,2\}$ is the *modality*; $a_1^2 \neq 1$ for $(s,k) = (4,1)$; $a_1 > 0$ for $s \neq k+3$; $a_1^2 \neq 1$ for $(s,k) = (6,2)$; $a_1 > 0$ in other cases.

Theorem 1. *Consider the G-right equivalence classes of the G-invariant function germs $F_{s,k}(x,y,0,\hat{a})$ in two variables at $(0,0)$, $k = 0,1,2$. These singularities have G-codimension k, G-Milnor number $k+m+1$, G-versal deformation $F_{s,k}(x,y,\lambda,a)+\lambda_0$, and form a complete list of G-invariant singularities with G-codimensions $k \leq 2$. The complement to their union in the space $\mathfrak{n}^2_G$ of G-invariant germs at $(0,0)$ having a critical point $(0,0)$ with a critical value 0, has codimension > 2 in $\mathfrak{n}^2_G$.*

Theorem 1 does not follow from the classification [1].

The proof of Theorem 1 is based on the following lemmas.

Lemma 1. *The algebra of G-invariant polynomials in x, y with real coefficients is multiplicatively generated by the homogeneous polynomials $|z|^2$, $\mathrm{Re}(z^s)$, $\mathrm{Im}(z^s)$. It is the linear span of the polynomials $\mathrm{Re}(z^k\bar{z}^l)$, $\mathrm{Im}(z^k\bar{z}^l)$, where $s \mid (k-l)$. In particular, any G-invariant function $f(z)$ has a Taylor series at $(0,0)$ of the form*

$$\mathrm{Re}\sum_{p,q\geq 0} c_{pq}|z|^{2p}z^{qs} = c_0 + c_1|z|^2 + \mathrm{Re}(c_2 z^s + c_3|z|^2 z^s) + c_4|z|^4 + \\ + c_5|z|^6 + c_6|z|^8 + \dots \qquad (1)$$

Lemma 2. *Let $f(x, y)$ be a G-invariant function with a Taylor series at $(0,0)$ of the form (1), where $s \geq 3$, $c_0 = 0$. The germ $f(x, y)$ at $(0,0)$ is G-right equivalent to $F_{s,k}(x, y, a, 0)$, $k = 0, 1, 2$, if and only if either*
(a) *$k = 0$, $c_1 \neq 0$, or*
(b) *$k = 1$, $c_1 = 0$, $c_2 \neq 0$ and $a = |c_4|/|c_2|^{4/s}$, or*
(c) *$k = 2$, $c_1 = |c_2|^2 - c_4^2 = 0$, $c_4 \neq 0$, $a = |c_5 - c_4\mathrm{Re}(\frac{c_3}{c_2})|/|c_2|^{3/2}$ for $s = 4$; $c_1 = c_4 = 0$, $c_2 \neq 0$, $a = |c_5|/|c_2|^{6/5}$ for $s = 5$; $c_1 = c_4 = 0$, $c_2 \neq 0$, $a_1 = \frac{|c_5|}{|c_2|^{6/s}}$, $a_2 = (c_6 - c_5\mathrm{Re}(\frac{6c_3}{sc_2}))/|c_2|^{8/s}$ for $s \geq 6$.*

Let us describe an application of Theorem 1 to the classification of structurally stable singularities of integrable systems with 2 and 3 degrees of freedom. Recall that *an integrable Hamiltonian system $(M, \omega, \mathcal{F})$ with n degrees of freedom* is given by a smooth mapping $\mathcal{F} = (f_1, \dots, f_n) : M \to \mathbb{R}^n$, where (M, ω) is a symplectic manifold, $\{f_i, f_j\} = 0$, $\dim M = 2n$.

Consider the action of the group G on the solid torus $V = D^2 \times S^1$ of the form $(z, \varphi_1) \mapsto (e^{2\pi i\frac{\ell}{s}}z, \varphi_1 + \frac{2\pi}{s})$, for $0 \leq \ell < s$, $(\ell, s) = 1$. Consider a cylinder $W = D^{n-1} \times (S^1)^{n-2}$ with coordinates $\lambda = (\lambda_i)_{i=1}^{n-1}$, $\varphi' = (\varphi_j)_{j=2}^{n-1}$.

One can deduce from Theorem 1 that singular $\mathbb{R}^n$-orbits of rank $n-1$ of typical integrable systems with $n = 2, 3$ degrees of freedom (such that $f_2, \dots, f_n$ generate a Hamiltonian action of $(n-1)$-torus [4]) have neighborhoods in which $\mathcal{F} : M \to \mathbb{R}^n$ is left-right equivalent to the *standard model* $(M_{\frac{\ell}{s}}, \mathcal{F}_{\frac{\ell}{s},k,a})$ of the form

$$M_{\frac{\ell}{s}} = (V/G) \times W, \quad \omega_{\frac{\ell}{s}} = \mathrm{d}x \wedge \mathrm{d}y + \sum_{j=1}^{n-1} \mathrm{d}\lambda_j \wedge \mathrm{d}\varphi_j,$$

$$\mathcal{F}_{\frac{\ell}{s},k,a} : M_{\frac{\ell}{s}} \to \mathbb{R}^n, \quad \mathcal{F}_{\frac{\ell}{s},k,a}(z, \varphi_1, \lambda, \varphi') = (\lambda, F_{s,k}(z, a(\lambda), \lambda')),$$

where $0 \leq k < n$, $\lambda' = (\lambda_1, \dots, \lambda_k)$, $a_i(\lambda)$ are smooth functions, $a_1(\lambda) \equiv 1$ for $(s,k) \notin \{(4,1),(6,2)\}$; $a_2(\lambda) \equiv 1$ for $(s,k) = (6,2)$.

The latter result for $n = 2$, $k = 1$ describes parabolic trajectories with resonances [2], whereas the result for $n = 3$, $k = 2$ describes their typical bifurcations (see [3, Example 3.6] and references therein).

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THE ORIGINAL VERSION OF THE KURAMOTO – SIVASHINSKY EQUATION AND ITS LOCAL ATTRACTORS

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The Kuramoto–Sivashinsky equation is one of the most well-known nonlinear equations of mathematical physics [1,2]. At present, this equation is considered not only in the specialized literature, but is also discussed in textbooks and monographs (see, for example, [3,4]).

It is usually written in one of the following two forms

$$u_t + u_{xxxx} + bu_{xx} + b_2(u_x)^2 = 0 \tag{1}$$

or

$$v_t + v_{xxxx} + bv_{xx} + b_2(v^2)_x = 0. \tag{2}$$

The second equation is obtained from the first after differentiation with respect to x and renaming $v = u_x$. Here above $b, b_2 \in R$.

The article [2] was one of the first in which the following partial differential equation appeared

$$u_t + u_{xxxx} + bu_{xx} - a_2u_x^2 - a_3(u_x^3)_x + \alpha u = 0. \tag{3}$$

In equation (3) $b, a_2, a_3, \alpha \in R$. This equation was obtained as a result of an asymptotic analysis of one of the problems of hydrodynamics, in which the Kolmogorov model was used.

In the second part of the article [2], noting that $0 < \alpha << 1$, and also that a_3 is also a sufficiently small constant, these terms were discarded and equation (1) supplemented with periodic boundary conditions was considered.

The report is supposed to present the results of bifurcation analysis in the vicinity of the zero equilibrium state, if equation (3) is supplemented with boundary conditions that are natural from the point of view of applications in hydrodynamics.

$$u_x(t, 0) = u_x(t, \pi) = u_{xxx}(t, 0) = u_{xxx}(t, \pi) = 0, \tag{4}$$

$$u(t, 0) = u(t, \pi) = u_{xx}(t, 0) = u_{xx}(t, \pi) = 0, \tag{5}$$

$$u_x(t, 0) = u_x(t, \pi) = u_{xx}(t, 0) = u_{xx}(t, \pi) = 0. \tag{6}$$

When analyzing the corresponding boundary value problems, we will assume that $\alpha = \alpha_0\varepsilon, \alpha_0 \in R_+, \varepsilon \in (0, \varepsilon_0), 0 < \varepsilon_0 << 1$.

An analysis of the stability of a homogeneous equilibrium state in the linear approximation for $\varepsilon = 0$ has shown that the equilibrium state $u = 0$ is asymptotically stable if

– $b < 1$ for boundary value problems (3), (4) and (3), (5);

– $b < 4$ for the boundary value problem (3), (6).

For all three boundary value problems, local bifurcations were studied if

– $b = 1 + \gamma_0\varepsilon, \gamma_0 \in R$ for boundary value problems (3), (4) and (3), (5).

Finally,

– $b = 4 + \beta_0\varepsilon, \beta_0 \in R$ for the boundary value problem (3), (6).

If we consider the boundary value problem (3), (6), then with an appropriate choice of b the question of local bifurcations can be reduced to the analysis of the normal form

$$z' = -\alpha_0 y + l_1 z^2,\ z' = \gamma z + l_2 z^3, \tag{7}$$

where $\gamma = (4\beta_0/3) - \alpha_0, l_1 = 5a_2/2, l_2 = -(5a_3 + 41a_2^2/144)$. In this case, each equilibrium state of the system of differential equations (7) corresponds to the equilibrium state of the boundary value problem (3), (6) with stability (instability) inheritance. For the boundary value problem (3), (4) we obtain a similar normal form, but, of course, with different coefficients in the corresponding equations.

Finally, for the boundary value problem (3), (5), the question of local bifurcations can be reduced to the analysis of the differential equation

$$y' = (\gamma_0 - \alpha_0)y + \frac{4a_2}{3\pi}y^2.$$

Various boundary value problems for equations (1) and (2) were considered in [5,6].

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PROPAGATION OF AN AUTOWAVE IN A MEDIUM WITH DISCONTINUOUS CHARACTERISTICS

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The following problem is considered

$$\begin{cases} \varepsilon^2 \dfrac{\partial^2 u}{\partial x^2} - \varepsilon^2 \dfrac{\partial u}{\partial t} := f\left(u, x, \varepsilon\right), \; x \in (-1; 1)\,, \; t \in (0; T]\,, \\ u_x(-1, t, \varepsilon) = u_x(1, t, \varepsilon) = 0, \quad u\left(x, 0, \varepsilon\right) = u^0\left(x, \varepsilon\right), \;\; x \in (-1; 1)\,, \end{cases}$$

where

$$f(u, x, \varepsilon) = \begin{cases} f^{(L)}(u, x, \varepsilon), & -1 \leq x \leq x_0, \; t \geq 0; \\ f^{(R)}(u, x, \varepsilon), & 0 \leq x_0 \leq 1, \; t \geq 0. \end{cases}$$

Assumption 1 Let the equation $f^{(L)}(u, x, 0) = 0$ has an isolated root $\varphi^{(-)}(x)$, on the segment $x \in [-1, x_0]$ and the equation $f^{(R)}(u, x, 0) = 0$ has three ordered isolated roots on the segment $\varphi^{(-)}(x) < q(x) < \varphi^{(+)}(x)$, $x \in [-1; 1]$, and the inequalities hold

$f_u^{(L)}(\varphi^{(-)}(x), x, 0) > 0$, $x \in [-1, x_0]$; $f_u^{(R)}(\varphi^{(\mp)}(x), x, 0) > 0$, $x \in [-1; 1]$.

In case of Assumption 1 a front form solution of the problem exists [1].

We investigate the two variants of this solution.

1. *Stabilization of moving front to the solution of stationary problem.*

$$\varepsilon^2 \frac{d^2 u}{dx^2} = f(u, x, \varepsilon), \quad -1 < x < 1; \quad u_x(-1) = u_x(1) = 0.$$

In this case we set the following additionally conditions.

$\varphi_1(x) < \varphi_2(x) < 0.5\left(\varphi_1(x) + \varphi_3(x)\right)$, $x \in [x_0, 1]$. That provides the increasing front moving left.

There exists a value $p_0 \in (\varphi^{(-)}(x_0), \varphi^{(+)}(x_0))$ that is the solution to the equation

$$\int\limits_{\varphi^{(-)}(x_0)}^{p_0} f^{(L)}(u, x_0, 0)du = \int\limits_{\varphi^{(+)}(x_0)}^{p_0} f^{(R)}(u, x_0, 0)du$$

and besides the inequality holds: $f^{(L)}(p_0, x_0, 0) > f^{(R)}(p_0, x_0, 0)$.

The last inequality is a stability condition for stationary solution [2].

2. *The front moving through media.*

In this case we set the following additionally conditions.

$0,5(\varphi^{(-)}(x) + \varphi^{(+)}(x)) < q(x) < \varphi^{(+)}(x), \quad x \in [-1; 1]$. That provides the increasing front moving right.

For all $p \in (\varphi^{(-)}(x_0), \varphi^{(+)}(x_0))$

$$\int\limits_{\varphi^{(-)}(x_0)}^{p} f^{(L)}(u, x_0, 0)du \neq \int\limits_{\varphi^{(+)}(x_0)}^{p} f^{(R)}(u, x_0, 0)du.$$

For these problems, an algorithm for constructing an asymptotic approximation of the solution was developed, and the existence of a moving front solution was proved. For the first problem, the stabilization of the front form solution to the solution of the stationary problem is also proved.

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EXISTENCE AND STABILITY OF A STATIONARY SOLUTION WITH A TWO-SCALE TRANSITION LAYER OF A SYSTEM OF TWO SINGULARLY PERTURBED SECOND-ORDER DIFFERENTIAL EQUATIONS WITH QUASIMONOTONITY CONDITIONS OF DIFFERENT SIGNS

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The paper investigates the existence and asymptotic stability of a stationary solution to an initial-boundary value problem

$$\varepsilon^4 \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = f(u, v, x, \varepsilon), \quad \varepsilon^2 \frac{\partial^2 v}{\partial x^2} - \frac{\partial u}{\partial t} = g(u, v, x, \varepsilon), \quad x \in (0; 1), \quad t > 0;$$
$$\frac{\partial u}{\partial x}(0, t, \varepsilon) = \frac{\partial u}{\partial x}(1, t, \varepsilon) = 0, \quad \frac{\partial v}{\partial x}(0, t, \varepsilon) = \frac{\partial v}{\partial x}(1, t, \varepsilon) = 0, \quad t > 0;$$
$$u(x, 0, \varepsilon) = u_{init}, \quad v(x, 0, \varepsilon) = v_{init}, \quad x \in [0; 1];$$

here ε is a small parameter.

This system is considered together with quasimonotonicity conditions

$$f_v(u, v, x, 0) > 0, \;\; g_u(u, v, x, 0) < 0$$

that must be met for all $x \in (0, 1)$, and all possible values of u and v.

A parabolic problem with such quasimonotonicity conditions is an system of activator-inhibitor type. The existence and stability theorems were proved using the asymptotic method of differential inequalities; for this, it is necessary to construct the upper and lower solutions. Due to such quasimonotonicity conditions, the algorithm for constructing them is more complicated than in previous works on the same topic.

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TRANSVERSE VIBRATIONS OF VISCOELASTIC BEAMS OF VARIABLE LENGTH, TAKING INTO ACCOUNT THE ACTION OF DAMPING FORCES

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The resonance characteristics of viscoelastic beams with moving boundaries using the Kantorovich – Galerkin method are examined in the article. The phenomenon of resonance and steady passage through resonance are analyzed. One-dimensional systems whose boundaries move are widely used in engineering [1–5]. The presence of moving boundaries causes considerable difficulties in describing such systems. Exact methods for solving such problems are limited by the wave equation and relatively simple boundary conditions. Of the approximate methods, the Kantorovich-Galerkin method described in [5] is the most efficient. However, this method can also be used in more complex cases. This method makes it possible to take into account the effect of resistance forces on the system, the viscoelastic properties of an oscillating object, and also the weak non-stationarity of the boundary conditions. The paper considers the phenomena of steady-state resonance and passage through resonance for transverse oscillations of a beam of variable length, taking into account viscoelasticity and damping forces. Performing transformations similar to transformations [5], an expression is obtained for the amplitude of oscillations corresponding to the n-th dynamic mode. Expressions are also obtained that describe the phenomenon of steady state resonance and the phenomenon of passage through resonance. The expression that determines the maximum amplitude of oscillations when passing through the resonance was numerically investigated to the maximum. The dependence of the beam oscillation amplitude on the boundary velocity, viscoelasticity, and damping forces is analyzed. The results of numerical studies allow us to draw the following conclusions:
- with a decrease in the velocity of the boundary, viscoelasticity and damping forces, the amplitude of oscillations increases;
- as the boundary velocity, viscoelasticity and damping forces tend

to zero, the oscillation amplitude tends to infinity; In conclusion, we note that the above results make it possible to carry out a quantitative analysis of the steady state resonance and the phenomenon of passage through the resonance for systems whose oscillations are described by the formulated problem.

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GENERALIZED MISHCHENKO–FOMENKO CONJECTURE FOR SINGULAR POINTS OF LIE ALGEBRAS

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Let $\mathfrak{g}$ be a Lie algebra, and let $\mathfrak{g}^*$ be the dual space. Consider the following structure on $\mathfrak{g}^*$:

$$\mathcal{A}_x(x) = (c_{ij}^k x_k),\ x \in \mathfrak{g}^*.$$

This tensor defines the Lie–Poisson bracket on $C^\infty(g^*)$. The functions $f \in C^\infty$, lying in the kernel of the Lie–Poisson bracket are called Casimir functions.

It is also possible to consider a similar structure which is called a bracket with a frozen argument:

$$\mathcal{A}_a(x) = (c_{ij}^k a_k),\ a, x \in \mathfrak{g}^*.$$

A Lie algebra is called completely integrable if over this Lie algebra there is a complete set of functions that are in involution. A complete set is considered to contain n functionally independent functions, where n is defined by the formula:

$$n = \frac{1}{2}(dim\ g + ind\ g).$$

The greatest practical interest is sets of polynomials.

- Mishchenko–Fomenko's conjecture (**proved**): on the dual space g^* of any Lie algebra g, there is a complete set of polynomials that are in involution.

- Generalized Mishchenko–Fomenko's conjecture: on the dual space g^* of any Lie algebra g there is a complete set of polynomials in bi-involution, i.e., a set that is simultaneously in involution with respect to $\mathcal{A}_x$ and $\mathcal{A}_a$.

The first conjecture was proved by Sadetov in 2004 [1], but the sets obtained by him did not always turn out to be in involution with respect to the bracket with a frozen argument. It is worth noting that when applying the general method for constructing sets in bi-involution (Mischenko–Fomenko's argument shift method), firstly, the resulting sets are not complete for all Lie algebras (as, for example, for semisimple Lie algebras), secondly, not for all values of the parameter a, set of functions are functionally independent.
There is a criterion of completeness of such sets for regular shifts:
Theorem 1. (Bolsinov criterion). *[2] Algebra of functions, generated by shifting by element a is complete if and only if there is a line $x + \lambda a$, which does not intersect the set of singular elements.*

In this report I will talk about generalization of this theorem:
Proposition 2. *Reformulation of the theorem from [2]. Let N_1 be the set of generators for the algebra of functions on surfaces $f_i = const$, $f_i \in ker\{\cdot,\cdot\}_a$, obtained by a shift by the element $a \in \mathfrak{g}^*$ consists of:*

$$\frac{dim\ \mathfrak{g} - dim(Ann(a))}{2}$$

functions, if and only if

1) there is $x \in \mathfrak{g}^*$ such that $\langle x, a\rangle$, without line λa consists only of regular elements,

2) $ind(Ann(a)) = ind\ \mathfrak{g}$.

From this follows method for constructing sets in bi-involution. Also in my report I will talk about necessary conditions for completeness of such sets, obtained by this method:

Theorem 3. (Construction of complete sets with respect to singular points). *Let for some $a \in \mathfrak{g}^*$ there is $x \in \mathfrak{g}^*$, such that*

1) there is $x \in \mathfrak{g}^*$ such that $\langle x, a\rangle$, without line λa consists only of regular elements,

2) $ind(Ann(a)) = ind\ \mathfrak{g}$.

then set of polynomials, generated by shifting by element a, can be complemented to a complete set in bi-involution by a set of polynomials in involution, such that there differentials are in $Ann(a)$.

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INTEGRABILITY OF GUIDING-CENTRE MOTION FOR CHARGED PARTICLES IN MAGNETIC FIELDS

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Guiding-centre motion is a one-parameter family of Hamiltonian systems of 2 degrees of freedom in a space with three position coordinates and one velocity coordinate. It represents the dynamics of the centre of gyro-rotation of a charged particle in a magnetic field under the adiabatic assumption of conservation of its magnetic moment. For magnetic confinement of charged particles, an ideal design of magnetic field is one for which guiding- centre motion is integrable (for all values of magnetic moment). By a version of Noether's theorem, this is equivalent to the Hamiltonian and symplectic form possessing a continuous symmetry. This is the case for any axisymmetric magnetic field, but the question arose back in 1982 whether there are any other possibilities, in particular for which the symmetry does not act on the velocity component. Such a field has been called quasi-symmetric, because the symmetry does not have to be an isometry (if it is an isometry and the orbits of the symmetry group are bounded then it has to be an axisymmetry). We obtain many necessary conditions on the field for quasi-symmetry, but the question whether there are any examples remains open. They would be an analogue of Kovalevskaya's famous non- axisymmetric integrable tops and would be hugely important for the design of fusion reactors to avoid the toroidal current that axisymmetry requires.

This is joint work with Josh Burby and Nikos Kallinikos.

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KÄHLER–EINSTEIN METRICS WITH SYMMETRIES AND INTEGRABLE STRUCTURES

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The existence of holomorphic and isometric immersions of a given Kähler manifold into a space of constant holomorphic sectional curvature can be considered as a classical and long-staying problem in complex differential geometry (cfr. e.g. [1, 2]). In this talk, we focus in particular on the open problem of the classification of Kähler–Einstein manifolds that can be Kähler immersed into a complex projective space endowed with the Fubini-Study metric. We will deal with such problem in the special case of Kähler metrics admitting symmetries of rotational type, leading to some integrable distributions allowing a classification of such metrics by means of a complex analogous of the Arnold–Liouville theorem [3].

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LOCAL NORMAL FORMS FOR KÄHLER METRICS ADMITTING C-PROJECTIVE VECTOR FIELDS

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A projective vector field is characterized by its local flow mapping geodesics to geodesics. In 1882 Sophus Lie asked to describe all metrics that admit such vector fields. The Kähler analogue is a c-projective vector field which maps J-planar curves to J-planar curves. Such c-projective vector field is called *essential* if it is not an affine vector field. The Existence of an essential c-projective vector field implies the existence of a non-proportional c-projectively equivalent metric which in turn gives rise to integrals of the geodesic flow. In complex dimension 2 metrics admitting an essential c-projective vector field have been described in [1]. We classify these metrics up to isomorphisms, thus obtaining normal forms.

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RESONANCES IN A SYSTEM OF THREE COUPLED DIFFERENCE AUTOGENERATORS

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Consider a mathematical model for a system of three coupled difference autogenerators (see, for example, [1]):

$$\frac{\partial^2 u_j}{\partial t^2} + 2\mu\frac{\partial u_j}{\partial t} + \mu^2 u_j = \frac{\partial^2 u_j}{\partial x^2}, \quad j = 1, 2, 3\,, \tag{1}$$

$$\left.\frac{\partial u_j}{\partial x}\right|_{x=1} = 0,\ u_j\big|_{x=0} + \varphi\left(K\left(u_j\big|_{x=1} + \alpha u_{j-1}\big|_{x=0}\right)\right) = 0,\ u_0 = u_3\,. \tag{2}$$

Here $u_j(t,x)$ are the normalized voltages in the lines of the corresponding generators, 2μ – line losses, $K > 0$ is the coefficient of amplification, and parameter $|\alpha| \ll 1$ characterizes connection between the generators. The function $\varphi(z) = z(1+z^2)^{-1/2}$ is obtained from the nonlinear characteristic of the amplifier as a result of shifting the working point to the origin. With some additional conditions, replacement $u_j = e^{-\mu t}\left(h_j\left(t + (x-1)\right) + h_j\left(t - (x-1)\right)\right)$ leads the boundary value problem (1)–(2) to a system of difference equations

$$\begin{cases} v_j(t+1) = w_j(t) \\ w_j(t+1) = -(1-\varepsilon)v_j(t) - \varphi(Kw_j(t)) + \nu\varphi'(Kw_j(t))\varphi(Kw_{j-1}(t)), \end{cases} \tag{3}$$

where $w_j(t) = h_j(t)\exp(1-\mu t)$, $v_j(t) = w_j(t-1)$, $j = 1, 2, 3$, $v_0 = v_3$, $w_0 = w_3$, $\varepsilon = 1 - \exp(-2\mu)$, $\nu = \alpha K$, and the value $2K\exp(-\mu)$ is again indicated as K. If $\varepsilon = 0$ and $0 < K < 2$ the matrix of the linear part of system (3) has a pair of eigenvalues $\lambda_{1,2} = \exp(\pm i\omega_0)$, where $\omega_0 = \arccos\left(-K/2\right)$, of multiplicity three, which lies on the unit circle of the complex plane.

In this work, we find such values of the parameter K when main resonances $e^{i\omega_0 j} = 1$, $j = 1, 2, 3, 4$ are realized [2]. It is shown that resonances of $1:3$ ($\omega_0 = 2\pi/3$) and $1:4$ ($\omega_0 = \pi/2$) occur for particular values of the parameter K. In these cases, normal forms of the systems are constructed and their dynamic properties are investigated.

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LOCAL DYNAMICS OF A SECOND-ORDER EQUATION WITH A DELAY AT THE DERIVATIVE

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Consider a second-order differential equation with delayed feedback, which is an implementation of the modified Ikeda equation with a time delay:

$$\varepsilon\frac{d^2y}{dt^2}+\frac{dy}{dt}+\delta y=F\left(\frac{dy}{dt}(t-\tau)\right). \tag{1}$$

Here ε and δ are small and proportional parameters $0<\varepsilon\ll 1$, $\delta=k\varepsilon$, τ is a delay parameter, real and positive. The function F is sufficiently smooth, such that $F(0)=0$. Thus, equation (1) has a zero equilibrium state. Let study the local dynamics in the vicinity of the equilibrium state,in the phase space $C^1_{[-1,0]}$. Note, that the problem under consideration is singularly perturbed.

The characteristic quasi-polynomial of the linearized at zero equation (1) has the form:

$$\varepsilon\lambda^2+\lambda+k\varepsilon=\lambda\beta_1 e^{-\lambda}. \tag{2}$$

It is shown that for $|\beta_1|<1$, the zero equilibrium state is stable, and for $|\beta_1|>1$, it is unstable. In critical cases $\beta_1=\pm1$, the characteristic equation has an infinite number of roots tending to the imaginary axis at $\varepsilon\to 0$. Thus, the critical cases have infinite dimension.

To study the behavior of solutions in β_1 close to -1, the solution of problem (1) is reduced in the case under consideration to a partial differential equation (3) with boundary conditions (4):

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\frac{\partial^2 V}{\partial^2 t} + kV - \frac{k^2}{2}J^2(V) + \beta V + \beta_2 J\left(U_1\frac{\partial V}{\partial t}\right) + \beta_3 J\left(\left(\frac{\partial V}{\partial t}\right)^3\right), \quad (3)$$

$$\int_0^1 V(\tau, t)dt = 0, \ V(\tau, t) \equiv -V(\tau, t+1). \quad (4)$$

Theorem 1. *Let $V_*(\tau, t)$ be a bounded with its derivatives solution (3) with boundary conditions (4), where $V_*(\tau, t) = \sum\limits_{-\infty}^{\infty} e^{\pi(2n+1)it}V_n(\tau)$, then*

$$y(t) = \varepsilon \sum_{-\infty}^{\infty} e^{iIm(\lambda_{n0}+\varepsilon\lambda_{n1}+\varepsilon^2\lambda_{n2}+\ldots)t}V_n(\varepsilon^2 t) + \varepsilon\frac{\beta_2}{k}\int_0^1 \dot{V}^2(t, \varepsilon^2 t)dt,$$

is asymptotic in the residual with an accuracy of $O(\varepsilon^3)$ uniformly over $t \geq 0$ by the solution of (1).

Problem (3), (4) is an analog of normal form. It's solution determine main terry of asymptotic approximate for solution of (1). To study the behavior of solutions in the case of a close $\beta_1 = 1$, the solution of the problem (1) is reduced in the case under consideration to the differential equation (5) and the partial differential equation (6) with boundary conditions (7):

$$W^\tau(\tau) = \left(\varepsilon\beta - \frac{k}{4}\right)W(\tau) - \frac{2}{3}\beta_2^2\sqrt{k}iW^2(\tau)\overline{W}(\tau) + \frac{3}{2}\beta_3\varepsilon^{\frac{1}{2}}kW^2(\tau)\overline{W}(\tau), \quad (5)$$

$$\frac{\partial V}{\partial \eta} = \frac{1}{2}\frac{\partial^2 V}{\partial^2 t} + kV - \frac{k^2}{2}J^2(V) + \beta V + \beta_3 J\left(\left(\frac{\partial V}{\partial t}\right)^3\right), \quad (6)$$

$$\int_0^1 V(\eta, t)dt = 0, \ V(\eta, t) \equiv V(\eta, t+1). \quad (7)$$

$J(V)$ denotes the primitive function V with a zero mean:

$$J^2(V) = J(J(V)), \ (J(V))'_t \equiv V.$$

For $\varepsilon\beta < \dfrac{k}{2}$, the solution of equation (5) tends to zero.

Theorem 2. *Let* $\varepsilon\beta < \frac{k}{2}$, *let* $V_*(\eta, t)$ *be a bounded with its derivatives solution (6), (7), and* $V_*(\eta, t) = \sum\limits_{-\infty}^{\infty} e^{2\pi nit} V_n(\eta)$, *then*

$$y(t) = \varepsilon^2 \sum_{-\infty}^{\infty} e^{iIm(\lambda_{n0}+\varepsilon\lambda_{n1}+\varepsilon^2\lambda_{n2}+\ldots)t} V_n(\varepsilon^2 t)$$

is asymptotic with respect to the residual with an accuracy of $O(\varepsilon^3)$ *uniformly over* $t \geq 0$ *solution (1).*

Theorem 3. *Let* $\varepsilon\beta > \frac{k}{2}$, *let* $W_*(\tau)$ *is a stable solution (5), then*

$$y(t) = \varepsilon^{\frac{1}{4}} (W_*(\varepsilon t) e^{i\sqrt{\varepsilon k} t} + \overline{W}_*(\varepsilon t) e^{-i\sqrt{\varepsilon k} t}) + O(\varepsilon^{\frac{1}{2}})$$

is a periodic stable solution (1).

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ANISOTROPIC SPIN GENERALIZATION OF ELLIPTIC MACDONALD-RUIJSENAARS OPERATORS

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We propose commuting set of matrix-valued difference operators in terms of the elliptic Baxter-Belavin R-matrix in the fundamental representation of GL_M. In the scalar case $M = 1$ these operators are the elliptic Macdonald-Ruijsenaars operators, while in the general case they can be viewed as anisotropic versions of the quantum spin Ruijsenaars Hamiltonians.

In scalar case Ruijsenaars proved that commutativity of the operators written in the form with arbitrary function is equivalent to the system of functional equations and the elliptic Kronecker function solves this system [1].

We show that commutativity of the operators for any M is equivalent to a set of R-matrix identities. The proof of identities is based on the properties of elliptic R-matrix including the quantum and the associative Yang-Baxter equations. As an application of our results, we introduce elliptic generalization of q-deformed Haldane-Shastry model.

The talk is based on joint work with Andrei Zotov [2 ,3].

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BI-HAMILTONIAN AND BI-QUANTUM STRUCTURE OF INTEGRABLE HIERARCHIES. NON-DEFORMATION QUANTISATION

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Traditional quantisation theories start with classical Hamiltonian systems with variables taking values in commutative algebras and then study their non-commutative deformations, such that the commutators of observables tend to the corresponding Poisson brackets as the (Planck) constant of deformation goes to zero. I am proposing to depart from dynamical systems defined on a free associative algebra $\mathfrak{A}$. In this approach the quantisation problem is reduced to the problem of finding of a two-sided ideal $\mathfrak{J} \subset \mathfrak{A}$ satisfying two conditions: the ideal $\mathfrak{J}$ has to be invariant with respect to the dynamics of the system and to define a complete set of commutation relations in the quotient algebras $\mathfrak{A}_{\mathfrak{J}} = \mathfrak{A}/\mathfrak{J}$ [1].

To illustrate this approach I will consider the quantisation problem for the Volterra family of integrable systems. In particular, I will show that odd degree symmetries of the Volterra chain admit two quantisations, one of them is a standard deformation quantisation of the Volterra chain, and another one is new and not a deformation quantisation. The periodic Volterra chain admits bi-Hamiltonian and bi-quantum structures [2]. The method of quantisation based on the concept of quantisation ideals proved to be successful for quatisation of stationary Korteweg-de–Vries hierarchies [3]. The Toda hierarchy also admits bi-quantum structures and non-deformation quantisation.

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ON A TRAVELLING WAVE TYPE PERIODIC FRONTS IN REACTION–DIFFUSION–ADVECTION EQUATIONS

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A new class of singularly perturbed parabolic periodic boundary value problems for Reaction–Diffusion–Advection equations including modular sorces is considered:

$$\begin{aligned}
&\varepsilon^2\frac{\partial^2 u}{\partial x^2} - \varepsilon\frac{\partial u}{\partial t} - \varepsilon A(x,t)\frac{\partial u}{\partial x} - \\
&\qquad\qquad - f(u,x,t,\varepsilon) = 0,\ x \in (-1,1),\ t \in R,\\
&\frac{\partial u}{\partial x}(-1,t,\varepsilon) = h^0(t),\ \frac{\partial u}{\partial x}(1,t,\varepsilon) = h^1(t),\ t \in R,\\
&u(x,t,\varepsilon) = u(x,t+T).
\end{aligned} \tag{1}$$

The present paper is devoted to the development of the results of the author and his students on new classes of periodic parabolic problems with transition layers (see [1]). The interior transition layer type asymptotics and a modification procedure to get asymptotic lower and upper solutions is proposed. By using such lower and upper solutions, we prove the existence of a periodic solutions with an interior layer, estimate the accuracy of its asymptotics and establish the asymptotic stability of this solutions.

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EXISTENCE OF CONTRAST STRUCTURES IN A PROBLEM WITH DISCONTINUOUS REACTION AND ADVECTION

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In the paper, a boundary value problem for a singularly perturbed reaction-diffusion-advection equation is considered in a two-dimensional domain in the case of discontinuous coefficients of reaction and advection, whose discontinuity occurs on a predetermined curve lying in the domain. Consider the boundary value problem

$$\begin{aligned} \varepsilon^2\Delta u =&\varepsilon(\mathbf{a}(u,x),\nabla u)+f(u,x,\varepsilon), x=(x_1,x_2)\in D,\\ \left.\frac{\partial u}{\partial \mathbf{n}}\right|_{\partial D} =&0, \quad x\in\partial D. \end{aligned} \tag{1}$$

Here D is a simply connected domain on the plane (x_1,x_2) with a smooth boundary ∂D, ε is a small parameter in the interval $(0;\varepsilon_0)$, $\varepsilon_0>0$, $\mathbf{a}(u,x)=(a_1(u,x),a_2(u,x))$ is a two-dimensional vector function; $\mathbf{n}$ is the outward pointing normal to the curve ∂D.

Let C_0 be a simple smooth closed curve lying entirely in the domain D and given in a parametric form using smooth functions $x_i=\hat{x}_i(\theta), i=1,2$, where $\theta\in[0,\theta_0)$ and θ_0 is the period of the function $\hat{x}(\theta)$. The curve C_0 divides the domain into two parts: $D^{(-)}$ is bounded by C_0, and $D^{(+)}$ is bounded by the curves C_0 and ∂D. Assume that the following condition hold:

Condition 1. The functions $\mathbf{a}(u,x),\ f(u,x,\varepsilon)$ can be represented in the form

$$\mathbf{a}(u,x)=\begin{cases} \mathbf{a}^{(-)}(u,x), & u\in I_u, \quad x\in\bar{D}^{(-)};\\ \mathbf{a}^{(+)}(u,x), & u\in I_u, \quad x\in\bar{D}^{(+)}; \end{cases} \tag{2}$$

$$f(u,x,\varepsilon)=\begin{cases} f^{(-)}(u,x,\varepsilon), & u\in I_u, \quad x\in\bar{D}^{(-)},\ \varepsilon\in[0,\varepsilon_0);\\ f^{(+)}(u,x,\varepsilon), & u\in I_u, \quad x\in\bar{D}^{(+)},\ \varepsilon\in[0,\varepsilon_0); \end{cases} \tag{3}$$

where I_u is the interval of variation of the function u, and $f^{(\pm)}(u, x, \varepsilon)$, $\mathbf{a}^{(\pm)}(u, x)$ are sufficiently smooth functions in the corresponding domains of their definition.

It is shown that under some other conditions this problem has a solution with a sharp internal transition layer localized near the discontinuity curve. For this solution, an asymptotic expansion in a small parameter is constructed, and also sufficient conditions are obtained for the input data of the problem under which the solution exists. The proof of the existence theorem is based on the asymptotic method of differential inequalities. It is also shown that a solution of this kind is Lyapunov asymptotically stable and locally unique. The results of the paper can be used to create mathematical models of physical phenomena at the interface between two media with different characteristics, as well as for development of numerical-analytical methods for solving singularly perturbed problems.

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ASYMPTOTIC INTEGRATION OF PERTURBED HEAT EQUATION

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We construct the asymptotics for solutions as $t \to \infty$ of the perturbed heat equation (see [1])

$$\frac{\partial u}{\partial t} = \Delta u + g(x)\frac{\sin \omega t}{t^{\rho}}u, \quad x \in \Omega, \quad t \geq t_0 > 0 \tag{1}$$

with initial condition $u(t_0, x) = \varphi(x)$ and Neumann boundary condition $\partial u/\partial \nu = 0$, $x \in \partial\Omega$. Here the function $u(t, x)$ is considered in the bounded domain Ω in $\mathbb{R}^m$ with smooth boundary $\partial\Omega$. Symbol $\partial u/\partial \nu$

denotes the outward normal derivative with respect to $\partial\Omega$. The real-valued functions $\varphi(x)$ and $g(x)$ belong to $L_2(\Omega)$, ω and ρ are positive parameters. Finally, Δ denotes the Laplace operator with respect to x.

To obtain the asymptotics for solutions of Eq. (1) we propose a method for constructing the asymptotics for weak solutions of the differential equation

$$\dot{u} = \big[A + G(t)\big]u, \quad t \geq t_0, \tag{2}$$

as $t \to \infty$, where $u \in \mathcal{B}$ and $\mathcal{B}$ is a complex Banach space. Here A is a closed linear operator with dense domain in $\mathcal{B}$. Moreover, operator A is a generator of a strongly continuous semigroup of linear bounded operators $T(t)$: $\mathcal{B} \to \mathcal{B}$ $(t \geq 0)$. Further, $G(t)$ $(t \geq t_0)$ is a family of linear bounded operators, acting from $\mathcal{B}$ to $\mathcal{B}$ and having the form

$$G(t) = B(t) + R(t). \tag{3}$$

In (3) a family of linear bounded operators $B(t)$ possesses the following property: the operator-valued function $B(t)$ is strongly measurable on every segment $[t_0, T]$, $T \geq t_0$, and$\|B(t)u\|$ tends to zero in an oscillatory manner as $t \to \infty$ for each $u \in \mathcal{B}$. Family of linear bounded operators $R(t)$ is also strongly measurable on every segment $[t_0, T]$, $T \geq t_0$, and, moreover, there exists a function $\gamma(t) \in L_1[t_0, \infty)$ such that $\|R(t)u\| \leq \gamma(t)\|u\|$ for each $u \in \mathcal{B}$. The necessity of studying equations having form (2), where the operator-valued function $G(t)$ is a certain parametrical perturbation with continuous spectrum, was mentioned, for example, in the remarkable monograph [2, p. 230].

Eq. (2) is considered under the following conditions imposed on the operator A:

(i) we can decompose $\mathcal{B}$ into a direct sum

$$\mathcal{B} = \mathcal{X} \oplus \mathcal{Y},$$

where the finite dimensional linear subspace $\mathcal{X}$ is a linear span of the generalized eigenvectors of the operator A, corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_N$ with zero real parts (with account of their multiplicities);

(ii) the closed linear subspace $\mathcal{Y}$ is invariant under the semigroup $T(t)$ and, moreover, for each $y \in \mathcal{Y}$ the inequality

$$\|T(t)y\| \leq Ke^{-\alpha t}\|y\|, \quad t \geq 0,$$

where $K, \alpha > 0$, holds.

The mentioned above conditions (i) and (ii) are the standard requirements of the centre manifold theory (see, e.g., [3]). By using the main ideas of this theory as well as the ideas of the certain variant of the averaging method from [4], we propose a method for constructing the asymptotic representations for weak solutions of Eq. (1) as $t \to \infty$.

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WAVE PACKET SPREADING IN DISORDERED SOFT ARCHITECTED STRUCTURES

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We study the dynamical and chaotic behavior of a disordered one-dimensional (1D) elastic mechanical lattice which supports translational and rotational waves (see e.g. [1]). The model used in this work is motivated by the recently reported experimental results for a highly deformable architected lattice composed of aligned LEGO crosses connected by flexible links [2]. This lattice is characterized by strong geometrical nonlinearities and potential coupling of two degrees of freedom (DoFs) per site. Although the linear limit of this model consists of a linear Fermi-Pasta-Ulam-Tsingou (FPUT) and a linear Klein-Gordon (KG) uncoupled lattices, using rotational initial excitations we evoke nonlinear coupling between the system's two DoFs. Our findings show that nonlinear coupling results in a rich wave-packet spreading behavior in the presence of strong disorder [3]. In the weak nonlinear regime, we observe energy spreading only due to the coupling of the two DoFs, which is in contrast to what is known for a single KG lattice where the spreading occurs due to chaoticity [4]. Additionally, for strong nonlinearities, we show that for the disordered system, a wave-packet is able to reach near ballistic behavior which is not the case either for disordered, nonlinear, 1D KG or the FPUT lattices. We also reveal that chaos is persistent during energy spreading, although its strength decreases in time [5]. This is quantified by the dynamics of the system's finite-time maximum Lyapunov exponent. Our results show that flexible, highly nonlinear mechanical lattices with coupled DoFs per lattice site provide a real physical platform to study and observe rich wave dynamics, which cannot otherwise be observed with classical uncoupled fundamental models [6].

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STABLE ARCS CONNECTING GRADIENT-LIKE DIFFEOMORPHISMS ON SURFACES

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In 1976, S. Newhouse, J. Palis, F. Takens [2] introduced the concept of a stable arc connecting two structurally stable systems on a manifold. Following [2], a smooth arc φ_t is called *stable* if it is an inner point of the equivalence class with respect to the following relation: two arcs φ_t, φ'_t are called *conjugate* if there are homeomorphisms $h : [0,1] \to [0,1]$, $H_t : M \to M$ such that $H_t\varphi_t = \varphi'_{h(t)}H_t, t \in [0,1]$, H_t continuously depend on t.

It was also established in [1] that all points of a regular stable arc are structurally stable diffeomorphisms, except for a finite number of bifurcation diffeomorphisms that do not have cycles, heteroclinic tangency and have one non-hyperbolic periodic orbit, which is the orbit of unfolding generically non-critical saddle node or flip.

For Morse-Smale diffeomorphisms given on manifolds of any dimension, there are known examples of systems that cannot be connected by a stable arc. In this regard, the question naturally arises of finding an invariant that uniquely determines the equivalence class of the Morse-Smale diffeomorphism with respect to the connection relation by a stable arc (*component of a stable isotopy connection*).

In the talk, we will present a criterion for belonging to one component of a stable connection for orientation-preserving gradient-like diffeomorphisms on the two-dimensional sphere S^2. A complete classification of the Palis diffeomorphisms introduced by him as a class of surface cascades included in a topological flow is given. The technique of constructing arcs uses topological methods for studying dynamical systems based on the transition to the space of wandering orbits.

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CRITERION FOR THE EXISTENCE OF A CONNECTED CHARACTERISTIC SPACE OF ORBITS IN A GRADIENT-LIKE DIFFEOMORPHISM OF A SURFACE

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The classical approach to the study of dynamical systems is to single out a dual attractor-repeller pair, which are attractive and repulsive sets for all other trajectories of the system. If it is possible to choose a dual pair of attractor-repeller in such a way that the space of orbits in their complement (the characteristic space of orbits) is connected, then this creates the prerequisites for finding complete topological invariants of the dynamical system. Along this path, in particular, a number of classification results for Morse-Smale systems were obtained. Thus, the complete topological classification of Morse-Smale 3-diffeomorphisms is essentially based on the presence of a connected characteristic space of orbits associated with the choice of a one-dimensional dual attractor-repeller pair.

Let $f : M^n \to M^n$ is a Morse-Smale diffeomorphism given on a closed connected n-manifold. Denote by Ω^0_f, Ω^1_f, Ω^2_f the set of sinks, saddles and sources of diffeomorphism f. For any (possibly empty) f-invariant set $\Sigma \subset \Omega^1_f$ such that $cl(W^u_\Sigma) \setminus W^u_\Sigma \subset \Omega^0_f$, let

$$A_\Sigma = \Omega^0_f \cup W^u_\Sigma,\ R_\Sigma = \Omega^2_f \cup W^s_{\Omega^1_f \setminus \Sigma}.$$

It follows from the work of [1] that A_Σ and R_Σ are attractors and repellers, which are called *dual*. In the monograph [2] the set

$$V_\Sigma = M^n \setminus (A_\Sigma \cup R_\Sigma)$$

is called *characteristic space*, and the orbit space $\hat{V}_\Sigma = V_\Sigma/f$ of the action of f on V_Σ is called *characteristic orbit space*.

There are a number of examples where a choice of a dual pair leads to a complete topological classification of some subset of Morse-Smale dynamical systems. In most cases, finding complete topological invariants is based on the existence of a connected characteristic space of orbits for the class of systems under consideration. For example, according to [3], for any 3-Morse-Smale diffeomorphism, the characteristic space of orbits constructed for the set of Σ saddle points with a one-dimensional unstable manifold is connected. This fact played a key role in obtaining a complete topological classification of such diffeomorphisms obtained in [3]. According to [1], any Morse-Smale diffeomorphism given on a manifold of dimension $n > 3$ also has a connected characteristic space of orbits. For Morse-Smale diffeomorphisms on a surface, this is not true in the general case.

Report is devoted to learning the existence of a connected characteristic space of orbits for gradient-like (without heteroclinic points) diffeomorphisms on surfaces. Let $f : M^2 \to M^2$ be such a diffeomorphism. Note that for the set Σ^1_f, the attractor $A_{\Sigma^1_f} = \Omega^0_f \cup W^u_{\Omega^1_f}$ is dual to the zero-dimensional repeller $R_{\Sigma^1_f} = \Omega^2_f$ and, therefore, is connected. Hence it follows that any pair of different stock points ω_1, ω_2 is connected by a simple path L_{ω_1,ω_2} consisting of closures of unstable saddle manifolds.

The fact is that the periodic point of the diffeomorphism f has a *positive orientation type* if all the eigenvalues of the differential of the mapping at this point are positive, otherwise we will call the orientation type *negative*. The main result of the work is the following theorem.

Theorem 1. A gradient-like diffeomorphism $f : M^2 \to M^2$ has a connected characteristic space of orbits if and only if any pair of stock points ω_1, ω_2 of negative orientation type belonging to different orbits of the diffeomorphism f is connected by a simple path L_{ω_1,ω_2} consisting of closures of unstable manifolds of saddle points of negative orientation type. In this case, the connected characteristic space of orbits is homeomorphic to the Klein bottle if the diffeomorphism f has at least one drain with a negative orientation type and is homeomorphic to a two-dimensional torus otherwise.

Since the orientation-preserving diffeomorphism has a positive type of all stock points, the immediate consequence of the theorem is the following result.

Theorem 2. [[4], Theorem 1.1] Any orientation-preserving diffeomorphism f on any orientable surface M^2 has a connected characteristic space of orbits homeomorphic to a two-dimensional torus.

As the following result shows, any surface admits gradient-like diffeomorphisms without a connected characteristic space of orbits.

Theorem 3. On an orientable surface of any kind, there is an orientation-changing diffeomorphism that does not have a connected characteristic space. A gradient-like diffeomorphism without a connected characteristic space exists on an undirectable surface of any kind.

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INSTABILITY OF CONTRAST STRUCTURES IN REACTION-DIFFUSION PROBLEMS IN CASE OF REACTION DISCONTINUITY

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We consider the following one-dimentional singularly perturbed reaction-diffusion problems that naturally arise in mathematical models describing physical phenomena at the interface of two media with different characteristics:

$$\varepsilon^2\frac{d^2u}{dx^2} = f(u,x,\varepsilon), \quad x\in(-1;1), \quad u(\mp 1,\varepsilon) = u^{(\mp)}, \tag{1}$$

$$\varepsilon^2\frac{d^2u}{dx^2} = g(u,x,\varepsilon) + \varepsilon g_1(u,x,\varepsilon), \quad x\in(-1;1), \quad u(\mp 1,\varepsilon) = u^{(\mp)}, \tag{2}$$

where ε is a small parameter.

Function $f(u,x,\varepsilon)$ has a discontinuity of the reactive term at an internal point of the segment on which the problem (1) is stated:

$$f(u,x,\varepsilon)=\begin{cases} f^{(-)}(u,x,\varepsilon), & u\in I_u,\ -1\le x\le x_0-0,\\ f^{(+)}(u,x,\varepsilon), & u\in I_u,\quad x_0+0\le x\le 1,\end{cases}$$

where $f^{(+)}(u,x,\varepsilon)\neq f^{(-)}(u,x,\varepsilon),\ x=x_0$, I_u is the interval of variation of the function u, $f^{(\pm)}(u,x,\varepsilon)$ are sufficiently smooth functions in their respective domains of definition.

The key feature of the secound problem (2) under consideration is a weak discontinuity of the reactive term:

$$g_1(u,x,\varepsilon)=\begin{cases} g_1^{(-)}(u,x,\varepsilon), & u\in I_u,\ -1\le x\le x_0-0,\\ g_1^{(+)}(u,x,\varepsilon), & u\in I_u,\quad x_0+0\le x\le 1,\end{cases}$$

where $g_1^{(+)}(u,x,\varepsilon)\neq g_1^{(-)}(u,x,\varepsilon),\ x=x_0,\ I_u$ is the interval of variation of the function $u,\ g_1^{(\pm)}(u,x,\varepsilon), g(u,x,\varepsilon)$ are sufficiently smooth functions in their respective domains of definition.

For these problems, conditions are obtained under which solutions with an inner transition layer appearing near the point x_0 are unstable. Existence theorems are proved using the matching method of asymptotic expansions. Asymptotic approximations are constructed using the Vasil'eva method. The proofs of the instability of solutions are based on the construction of an unordered pair of upper and lower solutions and on the application of a corollary of the Krein–Rutman theorem.

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INFINITE-COMPONENT KP HIERARCHY

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We introduce multidimensional Fermi fields and consider the generalized Kadomtsev-Petviashvili hierarchy associated with them and the generalized relativistic hierarchies of the Toda lattice obtained in a way similar to that proposed by the Kyoto school. This is a generalization two-dimensional Fermi fields introduced by O. Zaboronsky when describing the model of normal matrices. We show that the quantum TL equation at the point of a free fermion arises naturally. We will discuss the bosonization procedure.

(This is according to our old discussion with Oleg Zaboronsky. Unpublished).

INTEGRABLE SYSTEMS OF THE INTERMEDIATE LONG WAVE TYPE IN 2 + 1 DIMENSIONS

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We classify 2 + 1 dimensional integrable systems with nonlocality of the intermediate long wave type. Links to the 2 + 1 dimensional waterbag system are established. Dimensional reductions of integrable systems constructed in this paper provide dispersive regularisations of hydrodynamic equations governing propagation of long nonlinear waves in a shear flow with piecewise linear velocity profile (for special values of vorticities).

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KORTEWEG–DE VRIES EQUATION WITH POWER NONLINEARITY: SOLITONS, COMPACTONS AND ROGUE WAVES

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The analyze of the main properties of soliton solutions to the generalized Korteweg- de Vries equation $u_t + [F(u)]_{xx} + u_{xxx} = 0$, where the leading term is $F(u)$ qu^{α}, $\alpha > 0$ is done. The far field of such solitons may have three options. For $q > 0$ and $\alpha > 1$ the analysis re-confirmed that all travelling solitons have "light" exponentially decaying tails and propagate to the right. If $q < 0$ and $\alpha < 1$, the travelling solitons (so named compactons) have a compact support (and thus vanishing tails) and propagate to the left. For more complicated $F(u)$ and $\alpha > 1$ (e.g., the Gardner equation) standing algebraic solitons with "heavy" power-law tails may appear. If the leading term of $F(u)$ is negative, the set of solutions may include wide or table-top solitons (similar to the solutions of the Gardner equation), including algebraic solitons and compactons with any of the three types of tails. The solutions usually have a single-hump structure but if $F(u)$ represents a higher-order polynomial, the generalized Korteweg de Vries equation may support multi-humped pyramidal solitons.

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SELECTED RESULTS OF NUMERIC INTEGRATION APPROXIMATION IN THE PROBLEM OF N-BODY

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The evolution of the orbital elements of asteroids is obtained by numerical integration. After that, it is approximated by two methods: a partial Fourier series and a sum of trigonometric terms with arbitrary frequencies. The results are discussed.

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KNOT AS A COMPLETE INVARIANT OF MORSE-SMALE 3-DIFFEOMORPHISMS WITH FOUR FIXED POINTS

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Recall that a diffeomorphism $f : M^n \to M^n$ defined on a smooth closed connected orientable n-dimensional manifold $(n \geq 1)$ M^n , is called a *Morse-Smale diffeomorphism* if

1) its nonwandering set Ω_f consists of a finite number of hyperbolic orbits;

2) the manifolds W^s_p, W^u_q intersect transversally for any nonwandering points p, q.

If σ_1, σ_2 are different saddle periodic points of the Morse-Smale diffeomorphism for which $W^s_{\sigma_1} \cap W^u_{\sigma_2} \neq \emptyset$, then the intersection $W^s_{\sigma_1} \cap W^u_{\sigma_2}$ is called a *heteroclinic intersection* . In this case, the linear connected components of dimension 1 are called *heteroclinic curves.*

In [1] a complete topological classification of Morse-Smale diffeomorphisms on arbitrary closed 3-manifolds is obtained. However, being designed for a wide class of flows, the description of invariants occupies the main part of the classification. In some special cases, invariants are often found in a more natural way, without considering them as part of the generality. Thus, in this paper we establish that for a wide class of 3-diffeomorphisms whose nonwandering set consists of exactly four points, the equivalence class of the Hopf knot on the manifold $\mathbb{S}^2 \times \mathbb{S}^1$ is a complete invariant.

Recall that a *node* on a manifold $\mathbb{S}^2 \times \mathbb{S}^1$ is a smooth embedding $\gamma : \mathbb{S}^1 \to \mathbb{S}^2 \times \mathbb{S}^1$ or the image of this embedding $L = \gamma(\mathbb{S}^1)$. Knots γ, γ' are said to be *smoothly homotopic* if there exists a smooth mapping $\Gamma : \mathbb{S}^1 \times [0,1] \to \mathbb{S}^2 \times \mathbb{S}^1$ such that $\Gamma(s,0) = \gamma(s)$ and $\Gamma(s,1) = \gamma'(s)$ for any $s \in \mathbb{S}^1$. If, in addition, $\Gamma|_{\mathbb{S}^1 \times \{t\}}$ is an embedding for any $t \in [0,1]$, then the nodes are called *isotopic.* Knots L, L' are said to be *equivalent* if there exists a homeomorphism $h : \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^2 \times \mathbb{S}^1$ such that $h(L) = L'$. Let $[L]$ denote the equivalence class of the knot L.

A knot $L \subset \mathbb{S}^2 \times \mathbb{S}^1$ is called a *Hopf knot* if the homomorphism i_{L*} induced by the inclusion $i_L : L \to \mathbb{S}^2 \times \mathbb{S}^1$ is an isomorphism of the groups $\pi_1(L) \cong \pi_1(\mathbb{S}^2 \times \mathbb{S}^1) \cong \mathbb{Z}$.

Any Hopf knot is smoothly homotopic to the standard Hopf knot $L_0 = \{x\} \times \mathbb{S}^1$ (see eg [2]), but is not isotopic or equivalent to it in the general case. It is known from the results of [3] that Hopf knots are equivalent if and only if they are isotopic. B. Mazur [5] constructed a Hopf knot L_M that is nonequivalent and nonisotopic to L_0. In [3], countable families of pairwise nonequivalent Hopf knots are constructed.

Three-dimensional Morse-Smale diffeomorphisms with exactly four nonwandering points are divided into two classes, in one of which the diffeomorphisms have exactly one saddle point, in the other two. For diffeomorphisms of the first class, it was established in [6] that their topological conjugacy is completely determined by the equivalence of the Hopf nodes, which are the projection of a one-dimensional saddle separatrix. By virtue of [7], [6] any Hopf knot can be realized by a first-class diffeomorphism on the 3-sphere.

In this paper, we consider diffeomorphisms of the second class. Namely, we consider the class G of orientation-preserving Morse-Smale diffeomorphisms f defined on a closed manifold M^3 with the following properties:

- the nonwandering set of the diffeomorphism $f \in G$ consists of exactly four fixed points $\omega_f, \sigma_f^1, \sigma_f^2, \alpha_f$ with dimensions of unstable manifolds $0, 1, 2, 3$, respectively;
- the set $H_f = W^s_{\sigma_f^1} \cap W^u_{\sigma_f^2}$ is non-empty and path-connected (hence, it consists of the only non-compact curve)[1].

Let $f \in G$. Denote by ℓ_f^1, ℓ_f^2 the unstable separatrices of the point σ_f^1. Then (see, for example, [4]) the closure $cl(\ell_f^i)$ $(i = 1, 2)$ of the one-dimensional unstable separatrix of the point σ_f^1 is homeomorphic to a simple compact arc and consists from this separatrix and two points: the point σ_f^1 and the sink ω_f.

Let $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, $||\mathbf{x}|| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and $a : \mathbb{R}^3 \to \mathbb{R}^3$ is a diffeomorphism defined by $a(\mathbf{x}) = \frac{\mathbf{x}}{2}$. Define the mapping $p : mathbbR^3 \setminus O \to \mathbb{S}^2 \times \mathbb{S}^1$ by the formula

$$p(\mathbf{x}) = \left(\frac{x_1}{||\mathbf{x}||}, \frac{x_2}{||\mathbf{x}||}, \log_2(||\mathbf{x}||) \pmod 1 \right).$$

Let $V_{\omega_f} = W^s_{\omega_f} \setminus \omega_f$. Since the sink ω_f is hyperbolic, there exists a diffeomorphism $\psi_f : V_{\omega_f} \to \mathbb{R}^3 \setminus O$ conjugating the diffeomorphisms f and a. Let $p_{\omega_f} = p\psi_f : V_{\omega_f} \to \mathbb{S}^2 \times \mathbb{S}^1$ and $L_f^i = p_{\omega_f}(\ell_f^i)$, $i = 1, 2$.

Lemma 1. *For any diffeomorphism $f \in G$ the sets L_f^1, L_f^2 are equivalent Hopf nodes in $\mathbb{S}^2 \times \mathbb{S}^1$.* Denote by $\mathcal{L}_f = [L_f^1] = [L_f^2]$ the equivalence class of these knots.

Theorem 1. *Diffeomorphisms $f, f' \in G$ are topologically conjugate if and only if $\mathcal{L}_f = \mathcal{L}_{f'}$.*

Thus, the equivalence class of Hopf knots in $\mathbb{S}^2 \times \mathbb{S}^1$ is a complete topological invariant for diffeomorphisms of the Pixton class. Moreover, the following realization theorem holds.

Theorem 2. *For any equivalence class $\mathcal{L}$ of Hopf knots in $\mathbb{S}^2 \times \mathbb{S}^1$ there exists a diffeomorphism $f_{\mathcal{L}} : \mathbb{S}^3 \to \mathbb{S}^3 \in G$ such that $\mathcal{L}_{f_{\mathcal{L}}} = \mathcal{L}$.*

[1]It was proved in [8] that for diffeomorphisms of the second class the set H_f contains at least one non-compact heteroclinic curve, but in the general case it may contain more than one curve, including an infinite set of compact heteroclinic curves.

It follows directly from the theorems 1, 2 that the ambient manifold for diffeomorphisms of the class G is the 3-sphere $\mathbb{S}^3$.

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KNOT AS A COMPLETE INVARIANT OF DIFFEOMORPHISMS OF SURFACES WITH THREE PERIODIC ORBITS

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It is known that Morse-Smale diffeomorphisms with two hyperbolic periodic orbits exist only on the sphere and all of them are topologically conjugate to each other. However, if we admit the existence of three orbits, then the range of manifolds that admit them expands significantly. In particular, such orientation-preserving diffeomorphisms admit surfaces of any kind [1]. In the present paper, a complete invariant of topological conjugacy of Morse-Smale diffeomorphisms with three periodic orbits is found. It is completely determined by the homotopy type (a pair of coprime numbers) of a knot on the torus, which is the space of orbits of an unstable saddle separatrix in the space of orbits of the sink basin. With the help of the obtained result, it is possible to calculate the exact number of topological conjugacy classes of the considered diffeomorphisms on a given surface, as well as the relation of the genus of this surface to the homotopy type of the knot.

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NON-SINGULAR MORSE-SMALE FLOWS WITH THREE PERIODIC ORBITS ON ORIENTABLE 3-MANIFOLDS

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The topological equivalence of non-singular Morse-Smale flows under assumptions of various generalities is considered in a number of papers. However, in the case of a small number of orbits, the known invariants can be significantly simplified and, most importantly, the classification problem can be brought to realization by describing the admissibility of the obtained invariants. In a recent paper, an exhaustive classification of flows with two orbits on arbitrary closed n-manifolds was obtained. In this talk, a complete topological classification is obtained for flows with three periodic orbits given on orientable 3-manifolds.

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ALGEBRAIC ORIGINS OF INTEGRABILITY OF NONLINEAR EVOLUTION EQUATIONS

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Commutator identities on associative algebras generate solutions of linearized versions of integrable equations. We realize elements of associative algebra in a class of integral operators and develop a dressing procedure that enables to derive both: nonlinear integrable equations themselves and their Lax pairs, associated to the given commutator identities. Thus, the problem of construction of new integrable nonlinear evolution equations is reduced to the problem of construction of commutator identities on associative algebras. Different examples of commutator identities on such algebras are presented.

BEHAVIOUR OF THE SOLUTIONS OF A TRAFFIC FLOW MATHEMATICAL MODEL

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This paper presents an investigation of the traffic flow mathematical model, which describes the movement of $N \in \mathbb{N}$ vehicles. Model is represented by a system of delay-differential equations:

$$\begin{cases} \ddot{x}_n(t) = R_n\left[a_n\left(\dfrac{v_{max,n} - V_n}{1 + e^{k_n(-\Delta x_n(t,\tau)+s_n)}} + V_n - \dot{x}_n(t)\right)\right] + \\ \qquad +(1 - R_n)\left[q_n\left(\dfrac{\dot{x}_n(t)\,[V_n - \dot{x}_n(t)]}{\Delta x_n(t,\tau) - l_{n,\varepsilon}}\right)\right], \\ x_n(t) = \lambda_n, \quad \dot{x}_n(t) = v_n, \quad \text{for } t \in [-\tau, 0], \end{cases} \tag{1}$$

where $\Delta x_n(t,\tau) = x_{n-1}(t-\tau) - x_n(t)$ — distance between adjacent vehicles; $V_n = \min\left(\dot{x}_{n-1}(t-\tau), v_{max,n}\right)$ — speed function; τ — driver response time; $a_n > 0$ and $q_n > 0$ — coefficients describing the technical characteristics of vehicle acceleration and deceleration, respectively; $v_{max} > 0$ — maximum desired speed; $l_{n,\varepsilon} > 0$ — a safe distance corresponding $l_{n,\varepsilon} = l_n - \varepsilon$, where ε is an additive to prevent a vehicle from deceleration with infinite speed when $\Delta x_n(t, \tau)$ close enough to l_n; $k_n > 0$ and $s_n > 0$ — driver behavior parameters: k_n describes the smoothness of the vehicle driver movement adapting the speed to the speed of the one in front, and s_n describes the ceasing distance of the influence of the vehicle in front; λ_n — initial position of the vehicles; v_n — initial speed of the vehicles, and R_n — relay function as follows:

$$R_n = \begin{cases} 1, & \text{if } \Delta x_n(t,\tau) > \dfrac{\dot{x}_n^2(t)}{2\mu g} + l_n, \\ 0, & \text{if } \Delta x_n(t,\tau) \leq \dfrac{\dot{x}_n^2(t)}{2\mu g} + l_n, \end{cases}$$

where μ — coefficient of friction, and g — acceleration of gravity. Function R_n describes the "acceleration-deceleration" switch.

An analysis of the stability of a uniform driving mode was carried out for the model. This driving mode assumes that all vehicles are moving with the same speed v_{max} and at a distance $\Delta c_n = c_n - c_{n-1}$ from each other, where c_n is a decreasing sequence. For any decreasing sequence c_n a solution of the (1) system exists as follows:

$$x_n(t) = c_n + v_{max} t.$$

The stability of such a solution depends on the signs of expressions:

$$d_n = -\tau v_{max} + c_n - c_{n-1} - l_{n,\varepsilon}.$$

The following theorem is valid.

Theorem 1. *If the inequality $d_n > 0$ holds for $\forall n$, then the uniform mode is stable. If at least for one i inequality $d_i \leq 0$ is satisfied, then the uniform mode is unstable.*

It follows from the theorem that if all the vehicles of the flow move at a fairly large distance from each other, then this driving mode is stable. Stability is lost by increasing the speed v_{max}, the driver response time τ , the safe distance between vehicles l_n, or by reducing the distance between two adjacent vehicles Δc_n.

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ASYMPTOTICALLY STABLE NON-FALLING SOLUTIONS OF THE KAPITZA–WHITNEY PENDULUM

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The planar inverted pendulum with a vibrating pivot point in the presence of an additional horizontal force field is studied. We assume that the pivot point of the pendulum rapidly oscillates in the vertical direction and the period of these oscillations is commensurable with the period of horizontal force. This system can be considered as a strong generalization of the Kapitza pendulum. Previously it was shown [1] that for any horizontal force there always exists a non-falling periodic solution in the considered system. In particular, when there is no horizontal force, this periodic solution is the vertical upward position. In the talk we present analytical and numerical results concerning the existence of asymptotically stable non-falling periodic solutions in the system.

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ON THE EQUATION OF TETRAHEDRA, THE LOCAL YANG–BAXTER EQUATION AND SELF-DISTRIBUTIVE STRUCTURES

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Let V, W be some sets, $Y(\alpha) : V^2 \to V^2$ be a mapping depending on the parameter $\alpha \in W$. The notion of the local Yang–Baxter equation has been introduced by Maillet and Nijhoff in [1]. By the local Yang–Baxter equation we mean the equation

$$Y_{12}(\tilde{\alpha}) \circ Y_{13}(\tilde{\beta}) \circ Y_{23}(\tilde{\gamma}) = Y_{23}(\gamma) \circ Y_{13}(\beta) \circ Y_{12}(\alpha).$$

Here $\circ$ means superposition. Let $T : V^3 \to V^3$. By the equation of tetrahedra [2], we mean the following equation:

$$T_{123} \circ T_{145} \circ T_{246} \circ T_{356} = T_{356} \circ T_{246} \circ T_{145} \circ T_{123}.$$

Here $T_{ijk} : V^6 \to V^6$ acts on the ith, jth and kth components of an element from V^6 as T, and on the rest of the components are trivial. For example,

$$T_{246}(x, y, z, r, s, t) = (x, u(y, r, t), z, v(y, r, t), s, w(y, r, t)).$$

The paper establishes a connection between the solutions of the local Yang-Baxter equation and the solutions of the equation of tetrahedra of a special form. A construction is presented that makes it possible to obtain solutions of the tetrahedra equation associated with self-distributive structures. Some interesting examples of solutions obtained using this approach are considered.

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EXACT SOLUTION OF ONE COUNTABLE-DIMENSIONAL SYSTEM OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Let us consider the linear space of real two-sided sequences:

$$u = (\ldots, u_{-n}, \ldots, u_0, \ldots, u_n, \ldots), \qquad n \in \mathbb{Z}. \tag{1}$$

Further let one endow space (1) by the norm:

$$\|u\| = \sum_{n=-\infty}^{+\infty} |u_n| \tag{2}$$

and by the following operation of multiplication of two vectors u and v of this space:

$$(u \star v)_n = \sum_{k=-\infty}^{+\infty} u_k \, v_{n-k} \,, \tag{3}$$

then set of vectors (1) with finite norm (2) transforms to Banach algebra $l_1(\mathbb{R})$ [1].

The report deals with the next Cauchy problem for vector $u(x,t) \in l_1(\mathbb{R})$:

$$\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x} + u \star u = 0\,,\; u(x,0) = u^0(x) \in l_1(\mathbb{R})\,,\; x \in \mathbb{R}\,,\; c \in \mathbb{R}\,, \tag{4}$$

which is distributed generalization of the Cauchy problem for vector $u(t) \in l_1(\mathbb{R})$ described in work [2].

Using formula (3) one can rewrite equation (4) in components as countable-dimensional system of partial differential equations:

$$\frac{\partial u_n(x,t)}{\partial t} + c\,\frac{\partial u_n(x,t)}{\partial x} + \sum_{k=-\infty}^{+\infty} u_{n-k}(x,t)\, u_k(x,t) = 0\,,$$

$$u_n(x,0) = u_n^0(x)\,,\; n \in \mathbb{Z}\,. \tag{5}$$

It is easy to see that system (5) arises from the Cauchy problem for the following nonlinear integro-differential equation:

$$\frac{\partial u(x,y,t)}{\partial t} + c\,\frac{\partial u(x,y,t)}{\partial x} + \int_{-\infty}^{+\infty} u(x, y-\eta, t)\, u(x,\eta,t)\, d\eta = 0\,, \tag{6}$$

namely, quantization of plain $\mathbb{R}^2$ in y-direction with step δ generates from function $u(x,y,t)$ countable set of functions $u_n(x,t) = u(x, n\,\delta, t)$, $n \in \mathbb{Z}$. And after that change of integral in equation (6) by corresponding to it integral sum and further scaling $\delta\, u_n \to u_n$ ought to give rise to system (5) exactly.

Further let us introduce the generating function for the solution $u(x,t) \in l_1(\mathbb{R})$ of the Cauchy problem (5):

$$U(z; x, t) = \sum_{n=-\infty}^{+\infty} u_n(x,t)\, z^n\,, \tag{7}$$

then using the Laurent expansion (7) one can rewrite system (5) as follows:

$$\frac{\partial U(z;x,t)}{\partial t}+c\,\frac{\partial U(z;x,t)}{\partial x}+U^2(z;x,t)=0,\; U(z;x,0)=U^0(z;x)\,, \quad (8)$$

where function

$$U^0(z;x)=\sum_{n=-\infty}^{+\infty}u_n^0(x)\,z^n \qquad (9)$$

is the generating function for the initial condition of the Cauchy problem (5).

The Cauchy problem (8) can be easily solved in the framework of the well-known method of characteristics:

$$U(z;x,t)=\frac{U^0(z;x-c\,t)}{1+t\,U^0(z;x-c\,t)}\,. \qquad (10)$$

Comparing expressions (7) and (10) it is not difficult to find general representation of exact solution of the Cauchy problem (5):

$$u_n(x,t)=\frac{1}{2\,\pi\,i}\oint_{C_\rho}\frac{U^0(z;x-c\,t)}{1+t\,U^0(z;x-c\,t)}\,\frac{dz}{z^{n+1}}\,, \qquad (11)$$

integration along the circle $C_\rho=\{z\in\mathbb{C}\,|\,|\,z\,|=\rho\}$ being counter clockwise.

In practice instead of usage of formula (11) it is more suitable to derive the Laurent expansion (7) and to extract exact solution $u(x,t)\in l_1(\mathbb{R})$ of the input Cauchy problem (4) from formula (10) straightforwardly.

In this report such trick is demonstrated on initial condition corresponding to the generating function (9) chosen in the following form:

$$U^0(z;x)=a(x)+\frac{b(x)}{2}\left(z+\frac{1}{z}\right)\,, \qquad x\in\mathbb{R}\,. \qquad (12)$$

In conclusion of the report it is necessary to note that from formula (7) one can easily observe that if generating function $U(z;x,t)$ represents exact solution of the Cauchy problem (4) with initial condition representing by generating function $U^0(z;x)$ then for any $p=2,3,4,\ldots$ generating function $U(z^p;x,t)$ represents exact solution of the Cauchy problem (4) with initial condition representing by generating function $U^0(z^p;x)$.

In other words exact solution of the Cauchy problem (4) corresponding to initial condition (12) gives rise to the countable set of new exact solutions of the Cauchy problem (4) too. From point of view of radio engineering this procedure just coincides with procedure of upsampling in digital signal processing [3].

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THE CHAOS GAME IN THE HYPERBOLIC PLANE OF POSITIVE CURVATURE

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A hyperbolic plane $\widehat{H}$ of positive curvature is realized in the Cayley – Klein projective model in the ideal domain of the Lobachevskii plane and is adjacent to the Lobachevskii plane along an absolute oval curve γ. The first systematic description of the geometry of the plane is proposed in the works of the first author (for instance, in [1, 2]). We organize the Chaos game on various polygons of the plane $\widehat{H}$ and get analogues of fractals.

The fundamental group G of the plane $\widehat{H}$ consists of all projective automorphisms of the curve γ. Each transformation from the group G is an isometry. Unlike the fundamental group of the transformations of the Euclidean plane, the group G contains no similarity transformations. In this regard, the very concept of fractal as an object with the property of self-similarity requires a significant adjustment in the hyperbolic geometry. Each time we talk about fractal, we take into account only a certain property of the object, similar to the property of the known fractal in the Euclidean geometry. For example, the distant analogues of fractals, called tilings of the plane $\widehat{H}$, were obtained in [3]. Fractal tilings are interesting in that after every cutting of a certain part of the tiling, the remaining part is equal to the original one.

In the upcoming report, we will present the objects obtained as a result of the Chaos game on trihedrals of the plane $\widehat{H}$. As in the case of the Sierpinski triangle of the Euclidean plane, these objects can be obtain by iteration method too. The partition of the original trihedral in the plane $\widehat{H}$ and trihedrals obtained at each iteration occurs along curves containing the midpoints of edges of the trihedral. But in the hyperbolic case, these curves are not straight lines. Their degree depends on the iteration step, doubling at each subsequent step. We obtained the analytical setting of such curves for the first and second iteration steps and examined the curves. In the software that implements Chaos game, we used the formulas from [4] for calculate the coordinates of the segment midpoints in the canonical frame of the first type. We present here some results.

Let A be a point in the plane $\widehat{H}$. The transformation, in which each point M of the plane $\widehat{H}$ passes into the middle of the smaller of the segments AM, we called a *quasi-homothety* with *center* A and *coefficient* 1/2 and denoted by qH_A. If the line AM is elliptic and $A \perp M$, then the point M has two images under qH_A. Consequently, a quasi-homothety is not a bijection and does not belong to the group G.

Let σ be a curve in the plane $\widehat{H}$ and let $qH_A(\sigma) = \sigma_1$, $qH_A(\sigma_1) = \sigma_2$.

If σ is an elliptic line, then σ_1 is a hyperbola containing the absolute points A_1, A_2 of the polar line p_A of the point A with respect to the absolute curve γ. In this case σ_2 is a curve of degree four with two nodes at the points A_1, A_2.

If σ is a hyperbolic line other than the line p_A, then σ_1 is a bihyperbola of one sheet containing the absolute points of the lines σ and p_A.

If σ coincides with the line p_A, then σ_1 is an elliptic cycle with center A.

If σ is a parabolic line with an absolute point S, then σ_1 degenerates into a pair of hyperbolic lines with the common point S. The absolute points A_1, A_2 of the line p_A lie on σ_1 also.

By means computer comparison of images, we ascertain that arcs of the investigated curves σ_1 and σ_2 are edges of curvilinear trihedrals obtained at the first and, respectively, the second iteration step when constructing a fractal corresponding to the result of the Chaos game on a trihedral in the plane $\widehat{H}$.

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TO THE MODEL OF ASYMMETRIC PENDULUM

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The model of mathematic pendulum is widely used in the study of the mean motion resonance in celestial mechanics.(see for example [1] and references therein). The corresponding Hamiltonian consists of a Keplerian and a resonant part. Usually it is assumed that Keplerian part is the same. However the semimajor axis above and below of the resonance is not exactly the same. By this reason the circulation solutions above and below the resonance are different. Therefore it is necessary to study the model of an asymmetric pendulum. The consequences of this model are considered.

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COHERENT STRUCTURES IN THERMAL AND WEAKLY TURBULENT ENVIRONMENTS

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Regular coherent structures emerge in many nonintegrable nonlinear wave systems, where they can coexist with a background of random waves. Examples are solitary waves that emerge from a background of wave turbulence, and discrete breathers that coexist with thermal lattice vibrations. In my talk I discuss solitary waves interacting with random Rayleigh-Jeans distributed waves of a nonintegrable and noncollapsing nonlinear Schrödinger equation are studied. Two opposing types of dynamics are identified: firstly, the random thermal waves can erode the solitary wave, secondly, this structure can grow as a result of this interaction. These two types of behavior depend on a dynamical property of the solitary wave, namely its angular frequency, and on a statistical property of the thermal waves, the chemical potential. These two quantities are equal at a saddle point of the entropy that marks a transition between the two types of dynamics: high-amplitude coherent structures whose frequency exceeds the chemical potential grow, smaller structures with a lower frequency decay. Either process leads to an increase of the wave entropy. Numerical simulations verify this results.

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BRIGHT SOLITONS IN A (2+1)-DIMENSIONAL OCEANIC MODEL: DYNAMICS, INTERACTION AND MOLECULE FORMATION

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In this work, we consider the following integrable (2+1)-dimensional nonlinear model equation which arises in different perspectives to describe the dynamics of ion acoustic waves in a magnetized plasma, deep water oceanic rogue waves, optical wave propagation in Erbium-doped coherently excited resonant waveguides, optical Bragg grating fiber, etc.:

$$iq_t + aq_{xy} + 2ib(qq_x^* - q^*q_x)q = 0, \tag{1}$$

where $q(x, y, t)$ is envelope of the wave, x and y are spatial variables while t denotes the temporal variable and the symbol $*$ represents the complex conjugate. Here the coefficients a and b correspond to the dispersion and nonlinearity. The above model can also be viewed as a (2+1)D extension of the standard (1+1)D focusing nonlinear Schrödinger equation which appear in diverse fields of physics and show promising characteristics. First, we construct bright soliton solutions by using the powerful bilinearization method and explore their propagation characteristics. Next, we investigate the interaction dynamics of bright solitons for different choices of parameters which show head-on and oblique type elastic interactions (reappear unaltered after interaction and exhibits only a phase-shift) with explicit asymptotic analysis. Finally, we unravel the possibility of soliton molecule through velocity resonance mechanism which gives rise to breather formation. Especially, we discuss the influence of dispersion and nonlinearity in the dynamics, interaction and molecules. The obtained results will add significant information to the context of higher dimensional solitons.

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ON THE LEBESGUE MEASURE OF BIRKHOFF CURVE

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Recently author has researched rotation number [1] and Hausdorff dimension [2] additive properties for a Birkhoff curve and one has constructed different Birkhoff curves prime examples [3]. However the question of the Lebesgue measure of the Birkhoff curves have remained unresolved.

Birkhoff curves turn out to be nowhere dense atoms separating the plane being common boundary two or more then three regions. But this does not mean that there not exist Birkhoff curve of positive Lebesgue measure. For instance, there exist Cantor set of positive Lebesgue measure.

Suppose Birkhoff curve Υ separates on $\nu + 1$ invariant regions, such that ν regions have compact closure and let us consider invariant region G_i, $i \in \overline{1, \nu}$ with compact closure. It is clear that $\operatorname{mes} G_1 > 0$, and

$$\operatorname{mes} \Upsilon = \operatorname{mes} (\mathbb{E}^2 \setminus G_0) - (\operatorname{mes} G_1 + \ldots + \operatorname{mes} G_\nu).$$

Now let us construct the disjoint open circles sequence in G_i with common squares $\operatorname{mes} G_1 - \mu_n$ such, that $\operatorname{mes} \Upsilon < \mu_{n+1} < \mu_n(1 - \alpha/z_n)$, where $\alpha < 1$ and $\mu_1 = 1 - \alpha$, $z_1 = 1$, and integers $z_1 = 1$, z_2, $\ldots$, z_n, $\ldots$ form the increasing sequence. Therefore if the sequence $\sigma_{n+1} = \sigma_n + 1/z_{n+1}$ diverges then $\operatorname{mes} \Upsilon = 0$.

Theorem *If the sequence* $z_1 = 1$, z_2, $\ldots$, z_n, $\ldots$ *contains arithmetic progression then* $\operatorname{mes} \Upsilon = 0$ *in another case* $\operatorname{mes} \Upsilon > 0$.

This is direct corollary Szemeredi theorem [4].

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CATASTROPHES OF SOLUTIONS TO THE EQUATIONS OF AN ISENTROPIC GAS FLOW AND CATASTROPHES OF SOLUTIONS TO THE LINEAR WAVE EQUATION

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It is shown that the catastrophe germs of smooth mappings defining all three typical catastrophes of solutions to the system of equations of a one-dimensional isentropic gas flow coincide with germs of similar singularities of solutions to the linear wave equation. A hypothesis is put forward that a similar inheritance should take place for typical singularities of solutions to systems of equations of an isentropic gas in spatially non-one-dimensional cases as well.

The study was made in collaboration with B. I. Suleimanov.

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NUMERICAL INVESTIGATION OF SPATIOTEMPORAL CHAOS IN MULTIDIMENSIONAL HAMILTONIAN SYSTEMS

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We discuss various numerical approaches for studying the chaotic dynamics of multidimensional Hamiltonian systems. In particular, we focus our analysis on determining the characteristics of chaotic evolution of initially localized energy excitations in a Hamiltonian lattice model describing the disordered Klein-Gordon (DKG) oscillator chain in one spatial dimension. The linear modes of the system are exponentially localized by disorder and consequently, energy localization (usually called Anderson localization [1]) is observed in the absence of nonlinearity. On the other hand, nonlinear interactions result to the destruction of the initial energy localization, leading to the eventual subdiffusive spreading of wave packets in two different dynamical regimes (the so-called 'weak' and 'strong chaos' spreading regimes), which are characterized by particular power law increases of the wave packet's second moment and participation number [2-6].

Quantifying the strength of chaos through the computation of the maximum Lyapunov exponent (MLE, see for example [7] and references therein), we observe that the MLE exhibits power law decays, with different exponents for the weak and strong chaos, whose values are distinct from -1 seen in the case of regular motion [8-10]. The spatiotemporal evolution of the coordinates' distribution of the deviation vector used to compute the MLE (the so-called deviation vector distribution – DVD) reveals that chaos is spreading through the random oscillation of localized chaotic hot spots in the excited part of the wave packet [8-10]. Furthermore, the implementation of the SALI/$GALI_2$ chaos indicator [11-13] permits the efficient discrimination between localized and spreading chaos, with the former dominating the dynamics for lower energy values, for which the system is approaching its linear limit [14]. In addition, by computing the time variation of the fundamental frequencies of the motion of each oscillator in the lattice, i.e. the so-called frequency map analysis (FMA) technique [15-17], we reveal several characteristics of the dynamics for both the weak and strong chaos regimes [18]. More specifically, we find that in both regimes chaos is more intense at the central regions of the wave packet, where also the energy content is higher, while the oscillators at the wave packet's edges exhibit regular motion up until the time they gain enough energy to become part of the highly excited portion of the wave packet. Furthermore, we find that in the strong chaos regime the chaotic component of the wave packet is not only more extended than in the weak chaos one, but in addition the fraction of strongly chaotic oscillators is much higher.

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HYDRODYNAMIC ENVELOPE SOLITONS IN IRREGULAR SEA STATES

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Gravity water waves propagating in basins of great depth are known to be modulationally unstable with respect to long perturbations (Benjamin – Feir instability). This effect is captured by the nonlinear Schrödinger equation (NLSE) applicable for small amplitudes and long modulations. In planar geometry the NLSE is completely integrable by means of the Inverse Scattering Transform. Then envelope solitons represent the asymptotic solution of the initial-value-problem in infinite line. They are deeply related to the modulational instability, and the “nonlinear” spectrum of the IST is thought to better reflect the statistical properties of partly coherent waves, compared to the traditional Fourier transform.

A windowed IST transform (WIST) was suggested in [1] to recover the solitonic content in irregular waves under the assumption that their dynamics may be approximated by the NLSE (at least within a certain interval of time). Meanwhile, applicability of the approximate theory to realistic oceanic waves remained unclear and questionable, as typical storm waves violate assumptions of the NLS theory.

Surprisingly, the WIST was shown able to reconstruct parameters of isolated nonlinear groups rather accurately [2], even when they are short (i. e., the spectrum is not narrow) and contain steep waves (i.e., the nonlinearity is not weak). This approach was applied to the data of direct numerical simulations of the primitive equations of hydrodynamics in [3]. The observation of a wave group persisting for more than 200 periods in nonlinear unidirectional irregular water waves was identified as the intense envelope soliton with remarkably stable parameters. Most of the extreme waves occured on top of this group, resulting in higher and longer rogue wave events.

In the next step we show that a moderate wave directionality does not cancel soliton-like wave patterns, but rather reduces their persistence to a reasonable extent. Hence, the concept of hydrodynamics envelope solitons can be useful for marine science applications.

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CLASSIFICATION OF MATRIX SYSTEMS OF $P_2 - P_6$ TYPE WITH OKAMOTO INTEGRAL

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We study non-abelian systems of Painlevé type. For their derivation, we introduce auxiliary autonomous systems with the frozen independent variable and postulate their integrability in the sense of existence of non-abelian first integrals generalizing the Okamoto Hamiltonians. All non-abelian PVI-PII systems with such integrals are found. For each of them, isomonodromic Lax pairs are presented. A coalescence limiting scheme for non-abelian Painlevé systems is constructed.

OSCILLATORY MATRICES AND CLUSTER ALGEBRAS

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Totally positive matrices [1] form the basis of the modern intensively developing field of cluster algebras, have applications in integrable models of statistical physics, representation theory of quantum algebras, diophantine equations and many other fields. However, they arose in the context of the classical problem of small oscillations of linear elastic continuums [2]. The talk will be focused on the origin of these structures and some applications in the modern theory of electrical networks.

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SYNCHRONOUS COLLISIONS OF SOLITONS OF THE KORTEWEG – DE VRIES EQUATION: EXACT SOLUTIONS AND STATISTICAL PROPERTIES

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Synchronous collisions of solitons of the Korteweg–de Vries equation are considered as a representative example of the interaction of a large number of solitons in a soliton gas. Statistical properties of the soliton field are examined for a model distribution of soliton amplitudes. N-soliton solutions ($N \leq 50$) are constructed with the help of the original numerical procedure using the Darboux transformation and 100-digits arithmetic. It is shown that there exist qualitatively different patterns of evolving multisoliton solutions depending on the amplitude distribution. Collisions of a large number of solitons lead to the decrease of values of statistical moments (the orders from 3 to 7 have been considered). The statistical moments are shown to exhibit long intervals of quasi-stationary behavior in the case of a sufficiently large number of interacting solitons with close amplitudes. These intervals can be characterized by the maximum value of the soliton gas density and by "smoothing" of the wave fields in the integral sense. The analytical estimates describing these degenerate states of interacting solitons are obtained.

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ON A MODULI SPACE OF ADMISSIBLE PAIRS IN ARBITRARY DIMENSION: RECENT RESULTS

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Moduli spaces of stable vector bundles and compactifications of these moduli spaces are closely related with the Yang–Mills gauge field theory.

One of the reasons for the search for compactifications of moduli spaces for vector bundles by locally free sheaves only, is the Kobayashi–Hitchin correspondence. It opens up opportunities to apply algebraic geometrical methods to problems of differential geometrical or gauge theoretical setting when consideration of the moduli of connections in the vector bundle is replaced by consideration of the moduli for slope-stable vector bundles.

Usually moduli schemes for stable vector bundles on a projective algebraic variety are non-projective (and non-compact), and for application of algebraic geometrical techniques it is useful to include the scheme (or variety) of moduli for vector bundles into some appropriate projective scheme as its open subscheme. This problem is called a problem of compactification of a moduli space. There arises a natural question of looking for compactifications of moduli for vector bundles and of moduli for connections which admit expanding of the Kobayashi–Hitchin correspondence to both compactifications in whole.

We consider [1] a class of so-called admissible pairs $((\widetilde{S}, \widetilde{L}), \widetilde{E})$, each of which consists of an N-dimensional admissible scheme $\widetilde{S}$ of some class with a certain ample line bundle $\widetilde{L}$ and of a vector bundle $\widetilde{E}$. An admissible pair can be produced by a transformation (called a resolution) of a torsion-free coherent sheaf E on a nonsingular N-dimensional projective algebraic variety S to a vector bundle $\widetilde{E}$ on a certain projective scheme $\widetilde{S}$. There are [2] the notions of stability (semistability) for admissible pairs and of M-equivalence for admissible pairs in the multi-dimensional case. The stability (semistability) notion is naturally related with with the classical stability (semistability) for coherent sheaves under the resolution. Also the M-equivalence for semistable admissible pairs is also related naturally with the S-equivalence of coherent sheaves under the resolution.

We will discuss recent results concerning constructing a compactification of the moduli space of stable vector bundles by attachment of admissible semistable pairs and an ambient moduli space of semistable admissible pairs.

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EXISTENCE OF SOLUTIONS FOR THE SYSTEM OF TWO ODES WITH MODULUS-CUBIC NONLINEARITY

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As one of the natural developments of [1] the following boundary value problem is considered:

$$\varepsilon^4\frac{d^2u}{dx^2} = f(u,v,x,\varepsilon), \;\; \varepsilon^2\frac{d^2v}{dx^2} = g(u,v,x,\varepsilon), \;\; x \in (0,1),$$
$$u'(0) = u'(1) = 0, \;\; v'(0) = v'(1) = 0,$$

where $\varepsilon \in (0, \varepsilon_0]$ is a small parameter, $g(u,v,x,\varepsilon)$ similarly to [1] is assumed to be sufficiently smooth on $\mathbb{R}^2 \times [0,1] \times [0,\varepsilon_0]$, while the function $f(u,v,x,\varepsilon)$ is such that

$$f(u,v,x,\varepsilon) := \begin{cases} f^{(-)}(u,v,x,\varepsilon), & u < 0, v \in \mathbb{R},\, x \in [0,1],\, \varepsilon \in [0,\varepsilon_0]\,, \\ f^{(+)}(u,v,x,\varepsilon), & u \geq 0, v \in \mathbb{R},\, x \in [0,1],\, \varepsilon \in [0,\varepsilon_0]\,; \end{cases}$$

both $f^{(\mp)}(u,v,x,\varepsilon)$ are sufficiently smooth on respective enclosed domains — this means that f is no more than jump-discontinuous across $\{u=0\}\times\mathbb{R}\times[0,1]\times[0,\varepsilon_0]$.

The main focus of the author is set on steplike solutions with internal transitional layer. To guarantee the existence of such solutions the following hypothesises are considered:

(H_1) Each of the equations $f^{(\mp)}(u,v,x,0)=0$ is uniquely solvable with respect to u such that the functions $u=\varphi^{(\mp)}(v,x)$ are the isolated solutions to these equations on $\mathbb{R}\times[0,1]$, and the inequalities $\varphi^{(-)}(v,x)<0<\varphi^{(+)}(v,x)$,
$f_u^{(\mp)}(\varphi^{(\mp)}(v,x),v,x,0)>0$ hold.

(H_2) Denote $h^{(\mp)}(v,x)=g(\varphi^{(\mp)}(v,x),v,x,0)$. Each of the equations $h^{(\mp)}(v,x)=0$ is uniquely solvable with respect to v such that the functions $v=\psi^{(\mp)}(x)$ are the isolated solutions to these equations on $[0,1]$ and the inequalities
$\psi^{(-)}(x)<\psi^{(+)}(x)$, $h_v^{(\mp)}(\psi^{(\mp)}(x),x)>0$ hold.

(H_3) On respective sets $f_v^{(\mp)}(u,v,x,\varepsilon)<0$ and $g_u(u,v,x,\varepsilon)<0$.
...

(H_7^*) $f^{(-)}(0,v,x,0)>0>f^{(+)}(0,v,x,0)$ for all $(v,x)\in\mathbb{R}\times[0,1]$.

To prove the existence of the solution some changes are introduced into the standard monotone iterations from [2] due to emergence of non-trivial idiosyncrasies caused by discontinuity of the first kind in f with respect to the unknown function u, though the definition and the construction of the sub- and supersolution are identical to [1]. Those changes are not only technical in nature but are also related to the definition of a solution per se. By imposing increasingly stricter conditions with respect to f it is possible to narrow down whether the problem has a generalized (in terms of differential inclusions), strong (in $W_2^2(0,1)$) or quasiclassical (u has a single point of discontinuity of its second derivative on $(0,1)$) solution.

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IRREGULAR SOLUTIONS IN THE PROBLEM OF DISLOCATIONS IN A SOLID STATE

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Consider the [1] system describing dislocations in the crystal structure of a solid

$$m\ddot{y_n} + a\sin y_n = y_{n+1} - 2y_n + y_{n-1},\ (n = 1, \dots, N), \tag{1}$$

where m — is a positive coefficient, $y_n = y_n(t, x_n)$ — is the deviation of the $n-$th atom from the equilibrium position. For $y_n(t, x_n)$ at the extreme values of n the periodicity conditions $y_{N+1} = y_1, y_0 = y_N$. Instead of $y_n(t)$ we will consider a function of two variables $y(t, x_n)$, where x_n is the angular coordinate on some circle: $x_n = 2\pi n/N$. It is assumed that the number of interacting elements N is large or, what is the same – $0 < \varepsilon = 2\pi N^{-1} \ll 1$.

It is natural to pass to a continuous mass distribution (see, for example, [1, 2]), as a result of which we obtain (after obvious renormalizations and replacement of $a\sin y$ by a more general function $f(y)$)

$$\ddot{y} + f(y) = y(t, x+\varepsilon) - 2y + y(t, x-\varepsilon). \tag{2}$$

For definiteness, we assume that the periodic boundary conditions are satisfied

$$y(t, x+2\pi) \equiv y(t, x). \tag{3}$$

We take the nonlinear function $f(y)$ in the form

$$f(y) = ay + by^3 + \varphi(y),$$

where $a = a_0 + \varepsilon a_1, a_0 > 0$, and $\varphi(y) = o(|y|^3)$ for $y \to 0$. Consider the question of constructing the asymptotics of solutions in some sufficiently small neighborhood of the zero equilibrium state. We will study irregular solutions (2), (3), which are formed on asymptotically high ($\varepsilon \to 0$) modes.

Let us fix the parameter $\delta \neq 0$ arbitrarily and study the solutions (2), (3), ormed on the modes numbered

$$k = \pm(2\delta\varepsilon^{-1} + \theta + m), \quad m = 0, \pm 1, \pm 2, \dots, \tag{4}$$

where the quantity $\theta = \theta(\varepsilon) \in [0, 1]$ complements the term $2\delta\varepsilon^{-1}$ to an integer.

Let us denote $\gamma^2(\delta, a_0) = 4\sin^2(\delta) + a_0$. Note that for the roots of the characteristic equation we have the equality

$$\lambda_m = \pm i\Big[\gamma(\delta)\Big(1 + \frac{\varepsilon(\theta + m)\sin(2\delta)}{\gamma^2(\delta)}\Big) + o(\varepsilon^2)\Big]. \tag{5}$$

Let us formulate the main result. Let us introduce the notation $M(\varphi)$ – the mean value of the function x with respect to the space variable $\varphi(\tau, x)$:

$$M(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(\tau, x)dx.$$

Consider the boundary value problem

$$\begin{gathered} 2i\gamma(\delta)\frac{\partial\xi}{\partial\tau} - a_1\xi - R(\delta)\cdot\Big[\frac{\partial^2\xi}{\partial z^2} + 2i\theta\frac{\partial\xi}{\partial z} - \theta^2\xi\Big] = 3\xi(|\xi|^2 + 2M(|\eta|^2))\cdot b, \\ -2i\gamma(\delta)\frac{\partial\eta}{\partial\tau} - a_1\eta - R(\delta)\cdot\Big[\frac{\partial^2\eta}{\partial z^2} + 2i\theta\frac{\partial\eta}{\partial z} - \theta^2\eta\Big] = 3\eta(2|\xi|^2 + M(|\eta|^2))\cdot b, \end{gathered} \tag{6}$$

with periodic boundary conditions

$$\xi(\tau, z + 2\pi) \equiv \xi(\tau, z),\ \eta(\tau, z + 2\pi) \equiv \eta(\tau, z), \tag{7}$$

where $R(\delta) = \cos(2\delta) - \dfrac{1}{4}\gamma^{-2}(\delta)\sin^2(2\delta)$.

The connection between the solutions of the boundary value problem (6), (7) and the original boundary value problem (2), (3) establishes the following assertion. It will contain the sequence $\varepsilon_n \to 0$, which is determined by the condition $\theta(\varepsilon) = \theta_0$.

Theorem 1. Fix arbitrary parameters δ and $\theta_0 \in [0, 1)$. Let $\xi(\tau, z)$ and $\eta(\tau, z)$ — be solutions to the problem (6), (7) for $\theta = \theta_0$, bounded for $\tau \to \infty$, $x \in [0, 2\pi]$. Then there is a sequence $\varepsilon_n \to 0$, defined by the condition $\theta(\varepsilon) = \theta_0$, which is a function for $\varepsilon = \varepsilon_n$

$$\widetilde{y}(t, x, \varepsilon) = \varepsilon y_1(\tau, t, x) + \varepsilon^2 y_2(\tau, t, x),$$

satisfies the boundary value problem (2), (3) up to $O(\varepsilon^3)$. Here

$$\begin{aligned}
y_1 &= \xi_+(\tau, z_+) \exp(A_+) + \xi_-(\tau, z_-) \exp(A_-) + \bar{c}c,\\
y_2 &= f_+(\tau, t, x) \exp(A_+) + f_-(\tau, t, x) \exp(A_-) + \bar{c}c,\\
f_+(\tau, t, x) &= -2ib\gamma(\delta)\xi_+(\tau, z_+) \int\limits_0^{z_-} \left(|\xi_-(\tau, s)|^2 - M\left(|\xi_-(\tau, z)|^2\right)\right) ds,\\
f_-(\tau, t, x) &= 2ib\gamma(\delta)\xi_-(\tau, z_-) \int\limits_0^{z_+} \left(|\xi_+(\tau, s)|^2 - M\left(|\xi_+(\tau, z)|^2\right)\right) ds,\\
A_\pm &= i\varepsilon^{-1}(\delta + \varepsilon\Theta)x \pm i\left(2\sin\frac{\delta}{2} + \varepsilon\Theta\cos\frac{\delta}{2}\right) t,\\
z_\pm &= x \pm \varepsilon\gamma^{-1}(\delta)\frac{\sin(2\delta)}{2} t.
\end{aligned}$$

Thus, the boundary value problem (6), (7) plays the role of a normal form for the original boundary value problem (2), (3) and determines its rapidly oscillating solutions in space.

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LINEARIZATION BY MEANS OF A FUNCTIONAL PARAMETER

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Let us consider a Hamiltonian system in a neighbourhood of an equilibrium or a symplectic map in a neighbourhood of a fixed point in a phase space of dimension 2n. Let us assume that the system depends on a functional parameter, a function of n variables. We study the possibility to use this functional parameter to obtain a system conjugated to a linear system on an open set.

CROSSOVER SCALING FUNCTIONS IN THE ASYMMETRIC AVALANCHE PROCESS

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We consider the particle current in the asymmetric avalanche process on a ring. It is known to exhibit a transition from the intermittent to continuous flow at the critical density of particles. We present the exact expressions for the first two scaled cumulants of the particle current and explain how they were obtained in the large time limit $t \to \infty$ via the Bethe ansatz and a perturbative solution of the TQ-equation. The results are presented in an integral form suitable for the asymptotic analysis in the large system size limit $N \to \infty$. We demonstrate the scaling exponents for both the average current per site and the diffusion coefficient in subcritical, critical and superscitical regimes. Also, we identify the crossover regime and obtain the scaling functions for the uniform asymptotics unifying these three regimes.

The talk is based on our work with Alexander Povolotsky, J. Phys. A: Math. Theor. 55 025202, arXiv: 2109.06318.

INTEGRABLE METRICS WITH HIGHER ORDER INVARIANTS ON THE SPHERE, ELLIPSOID, HYPERBOLOID AND PLANE

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We will discuss how to apply various symplectic maps and Maupertuis principle to construct new integrable metrics (geodesic flows) on the surfaces in Euclidean space. As an example, we will present a few integrable metrics on 2D sphere, ellipsoid, hyperboloid and plane with cubic, quartic and sextic invariants.

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ASYMPTOTIC APPROXIMATION OF BOUNDARY CONTROL PROBLEM FOR BURGERS-TYPE EQUATION WITH MODULAR ADVECTION

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The concept of asymptotic solution of the inverse problems for reaction-diffusion-advection equations applied to a new class of Burgers-type equation with modular advection, which has a time-periodic solution of moving front type. Asymptotic approximation for the solution of the boundary control problem was obtained.

These results will be illustrated by the following problem

$$\begin{aligned}
&\varepsilon\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial t}+\frac{\partial |u|}{\partial x}-q(t)\cdot u=0, \qquad x\in(-1,1); \quad t\in\mathbb{R},\\
&u(-1,t;\varepsilon)=u^{(-)}(t), \qquad u(1,t;\varepsilon)=u^{(+)}(t), \qquad t\in\mathbb{R},\\
&u(x,0;\varepsilon)=u(x,t+T;\varepsilon), \qquad x\in[-1,1], \qquad t\in\mathbb{R},
\end{aligned} \tag{1}$$

where ε – small parameter ($0 < \varepsilon \ll 1$), functions $q(t)$, $u^{(-)}(t)$ and $u^{(+)}(t)$ are sufficiently smooth and T-periodic in t, $q(t) \geq q_0 > 0$. Some application are in [1,2].

The boundary control problem is posed within the framework of the existence theorem of a solution to a direct problem and consists in determining one of the boundary conditions, at which the front will move according to a given time law, or boundary function must be constructed using the observed moving front location.

Asymptotic solution of the boundary control problem means the determining the boundary conditions under which the required law of front motion is achieved with a given accuracy. Sufficient conditions which guarantee the required periodic regime have been obtained. A class of reaction-diffusion-advection problems is highlighted for which an asymptotic solution of the boundary control problem can be obtained explicitly. It is shown that for the class of equations under consideration, the boundary control problem is reduced to algebraic relations linking, with a given accuracy, the observed parameters of the moving front with the coefficients in the equation and boundary conditions. Thus, if the law of front moving is given on a certain time interval (period), then the problem of determining the boundary condition that guarantee realization of the required mode with a given accuracy is reduced to a set of simple algebraic operations.

This approach can be applied to some other classes of the inverse problems for reaction-diffusion-advection equations with boundary and internal layers [3,4].

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ON SYMMETRIES OF QUAD SYSTEMS OF DIFFERENCE EQUATIONS

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Symmetries provide arguably the most reliable means to test and prove the integrability of a given equation. They are used in the analysis and classification of partial differential equations and differential-difference equations since 1970's and 1980's, and only very recently this approach has been extended to partial difference equations. In this talk I will consider a class of systems of partial difference equations in two dimensions and discuss the general form of their symmetries. I will derive necessary determining equations for symmetries by exploiting various dynamical and eliminable variables. Finally I will discuss how these equations yield not only the symmetries but conservation laws as well.

WAVE–PARTICLE RESONANT INTERACTIONS IN SPACE PLASMA – EFFICIENCY STUDY

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The theoretical and numerical study of the resonant wave-particle interactions is presented. Using the numerical analysis of the second order nonlinear, nonstationary differential equation for the wave phase on the particle trajectory, the efficiency of the relativistic electron energy gain is studied. An evaluation for energy gain during the stage of the energy exchange between the wave and the particle is given. The numerical experiments were performed for several sets of differential equation initial parameters. From the analysis provided it follows that

the acceleration process increases the particle initial energy up to three orders of magnitude. The particle velocity components, momentum dynamics, and phase plane structure for the acceleration time interval obtained in numerical simulations are shown. The efficiency of the acceleration process during the resonant interactions is discussed. The calculation results are presented in graphical forms. Conclusions about electron energy gain during the resonant wave-particle interactions in space plasmas are made.

A PEDAGOGICAL APPROACH TO LIMIT CYCLES

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Oscillations are states of "dynamical (in)stability", as opposed to the motionless (in)stability of simple equilibrium points. In this lecture, I will present, in a pedagogical way, a topic which is of great mathematical interest, but also has numerous and extremely important applications in physics, chemistry, biology and engineering: It concerns the so–called *limit cycles*, or *relaxation oscillations* occurring in a wide variety of non-conservative dynamical systems of dimension $d \geq 2$. I will start with simple examples of two competing species in 2 dimensions (the Lotka Volterra system [1,2] and examine the appearance of limit cycles numerically and theoretically [2,3] in systems of $d = 2, 3$ and dimensions in chemical [4], physical [5], and biological [1,2] models, in which the presence of a fixed point plays a crucial role. Finally, I will discuss some more recent results, where limit cycles occur in the *absence* of fixed points, in a class of dynamical systems with dimension $d \geq 4$ [6-8].

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STATISTICAL MECHANICS OF BEAM THERMALIZATION IN MULTIMODE OPTICAL FIBERS

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Many intriguing nonlinear effects have been recently discovered in multimode optical systems. Among them, we are interested here in spatial beam self-cleaning (BSC), whose underlying physical mechanism has so far remained controversial [1]. In the linear wave propagation regime, each fiber mode propagates with its own phase velocity. As a result, a multimode laser beam emerges from the fiber as speckled spatial intensity pattern. Whereas in the nonlinear regime, a spectacular reshaping of the output beam occurs, leading to a stable, and bell-shaped intensity profile with much increased brightness and beam quality. This effect is the manifestation of a nonlinearity-induced mechanism, which leads to the irreversible redistribution of the input beam energy among the multiple fiber modes.

Although BSC has been confirmed by many experiments by different groups and in various settings, multiple explanations have been given about its origin: nonreciprocal quasi-phase matched four wave mixing [1], self-organization [2], wave thermalization [3], condensation [4] and 2D hydrodynamic turbulence [5].

In recent works, we have demonstrated that BSC can be well described in terms of classical statistical mechanics and associated conservation laws. In short, the evolution of multimode waves can be predicted by the laws that rule a gas of particles, where nonlinearity takes the role of particle collisions that exchange energy among them. We carried out an extensive series of experiments, based on a holographic mode decomposition of the output beams from multimode fibers. The experiments demonstrate that BSC results from a thermalization of the multimode population, which is described by a Rayleigh-Jeans (RJ) distribution [6]. It is particularly interesting to consider the consequences of the conservation of the orbital angular momentum (OAM) of light, leading to a generalized RJ distribution [7].

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