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P.G. Demidov Yaroslavl State University

Second International Conference on Integrable Systems & Nonlinear Dynamics

2020

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Центр интегрируемых систем

ТЕНЗОР

Тензор

P. G. Demidov Yaroslavl State University
Centre of Integrable Systems
IT company “Tensor”

Second International Conference on Integrable Systems & Nonlinear Dynamics ISND–2020

ABSTRACTS

Yaroslavl, October 19–23, 2020

Ярославский государственный университет
им. П. Г. Демидова

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IT компания “Тензор”

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The P.G. Demidov Yaroslavl State University is hosting an international scientific conference on Integrable Systems and Nonlinear Dynamics from 19 to 23 October 2020 in the city of Yaroslavl. This is a collection of abstracts of the conference talks. The abstracts published here are edited by the authors.

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ON A HIERARCHY OF MULTI-COMPONENT GENERALISATION OF MKDV TYPE EQUATIONS

P. Adamopoulou

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In this talk I will discuss the construction of the hierarchy, based on the Drinfeld-Sokolov scheme, associated to a generalisation of modified KdV equation. The recursion operator and conserved densities for the hierarchy will be presented. Finally, I will present the soliton and breather solution for the whole hierarchy, obtained via the Darboux-Dressing method.

ESTIMATION OF COMPLETE ELLIPTIC INTEGRAL OF THE SECOND KIND BY MEANS OF FEATURES OF THE PASCAL SNAIL

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Complete elliptic integral of the second kind:

$$\mathbf{E}(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2\varphi}\, d\varphi\,, \qquad k \in [0,1]\,, \tag{1}$$

has often been arising in different branches of mathematics and its applications (see [1] and references there in) hence estimation of special function (1) via elementary functions is very important. To realize this task the following theorem has been proven.

Theorem. If $k \in [0,1]$ then the next inequality for function (1) is valid:

$$\mathbf{E}(k) \geq \frac{\pi}{4\sqrt{2}}\, \frac{\sqrt{k^4 + 2\,(1+\sqrt{1-k^2})^4}}{1+\sqrt{1-k^2}}\,. \tag{2}$$

Proof. Let one consider the Pascal snail in polar coordinates:

$$r = \cos\theta + b\,, \qquad \theta \in [0, 2\,\pi], \qquad b > 1\,. \tag{3}$$

Curve (3) bounds the following area:

$$S = \frac{\pi\,(1 + 2\,b^2)}{2}. \tag{4}$$

On the other hand length L of curve (3) is expressed via function (1) as follows:

$$L = 4\,(1 + b)\,\mathbf{E}(k), \tag{5}$$

where modulus of complete elliptic integral of the second kind is equal to:

$$k = \frac{2\,\sqrt{b}}{1 + b}\,. \tag{6}$$

Substituting expressions (4) and (5) into the well-known isoperimetric inequality $L^2 \geq 4\,\pi\,S$ and expressing parameter b of the Pascal snail in formula (6) via modulus k one obtains inequality (2).

Lower bound (2) for function (1) proves to be much better then estimations of complete elliptic integral of the second kind obtained both geometrically in work [1] and analytically in work [2].

This work has been done jointly with A.E. Rassadin.

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HOMOCLINIC POINTS IN 2-D AND 4-D GENERALIZED HENON MAPS

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We engage ourselves with the computation of discrete breathers in coupled 1D Hamiltonian particle chains. To do this, we compute the homoclinic intersections of invariant manifolds of a saddle point located at the origin of a class of 2N -dimensional invertible maps. We apply the parametrization method to express these manifolds analytically, as finite series expansions, and compute their intersections numerically to high precision. We first carry out this procedure for a two-dimensional family of generalized Hénon maps, prove the existence of a hyperbolic set in the non-dissipative case and show that it is directly connected to the existence of a homoclinic orbit at the origin. Introducing dissipation we demonstrate that a homoclinic tangency occurs beyond which the homoclinic intersection disappears. We then use the same approach to accurately determine the homoclinic intersections of the invariant manifolds of a saddle point at the origin of a 4D map consisting of two coupled 2D cubic Hénon maps. For small values of the coupling we are able to determine the homoclinic intersection, which ceases to exist once a certain amount of dissipation is present.

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NON-STANDARD LIOUVILLE TORI AND CAUSTICS IN PROBLEM OF LONG WAVES TRAPPED BY A SHORE

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We study asymptotic eigenfunctions of the two-dimensional wave operator $\hat{L} = \nabla D(x_1, x_2)\nabla$ in a domain Ω with the coefficient $D(x)$ degenerating on the boundary $\partial\Omega$. From the physical point of view, this problem describes standing waves trapped by beaches and islands in the long wave approximation.

This stationary problem is quite unusual, for instance, compared with bound states of Schrödinger operator. Indeed, the Hamiltonian system associated with the problem determines a geodesic flow with degeneracy near the boundary $\partial\Omega$. As a result, assuming that the flow is integrable, the level set of two first integrals may have the following pathologies near $\partial\Omega$. First, it is non-compact, and, second, it consists of trajectories whose momenta go to infinity at finite time.

However, using the completed phase space near the boundary allows one to compactify the first integrals level sets and obtain tori (called non-standard Liouville tori). The projection of these tori onto the configuration space may have singularities such as standard caustics (as in case of standard Liouville tori) and non-standard caustics (that project onto the boundary). Locally, in a neighborhood of caustics the studied asymptotic solutions can be represented in terms of Airy function (for a standard caustic) or Bessel function (for a non-standard one). Our main result [1] is that we construct the global uniform asymptotic formula for the solution in terms of Airy and Bessel functions. We also show that these formulas are quite efficient in a sense that they can be quite easily numerically implemented using software like Wolfram Mathematica, Maple, etc.

The work was supported by the Russian Science Foundation (project No. 19-11-13042).

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LOCAL AND GLOBAL DYNAMICS IN 1-D HAMILTONIAN LATTICES: FROM PHYSICS TO ENGINEERING

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Local and global stability properties of 1-D Hamiltonian lattices of N interacting particles have been studied extensively for more than 60 years, in view of their important implications for statistical mechanics and solid state physics [1]. Most studies so far have focused on analytic interparticle interactions, ranging from nearest neighbor to full range, often in the presence of on-site potentials [2]. Moreover, under periodic driving at one end of the lattice, the phenomenon of energy supratransmission in such lattices has been observed and thoroughly documented [2,3]. In the present lecture, I will first describe an approach from local to global dynamics and statistics in these systems based on the study of some of their simple periodic solutions (nonlinear normal modes) as the total energy is increased. Next, I will apply this approach to study analogous phenomena in 1-D Hamiltonian lattices that arise in various mechanical engineering applications, such as graphene elasticity, Hollomons law of work hardening and hysteretic damping. These involve nearest-neighbor interactions that are: (a) either purely non-analytic, (b) harmonic plus non-analytic or (c) analytic with non-analytic hysteretic damping effects [4,5]. In some cases, I will apply periodic driving at one end of the lattice and demonstrate the occurrence of energy supratransmission for the first time in these systems.

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DYNAMICAL SYSTEMS ON 2-TORUS, MODEL OF JOSEPHSON JUNCTION AND ISOMONODROMIC FAMILIES OF LINEAR SYSTEMS

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In 1973 B.Josephson received Nobel Prize for discovering a new fundamental effect in superconductivity concerning a system of two superconductors separated by a very narrow dielectric (this system is called the *Josephson junction*): there could exist a supercurrent tunneling through this junction. We will discuss a mathematical model of the overdamped Josephson junction given by a family of first order non-linear ordinary differential equations that defines a family of dynamical systems on two-torus $\mathbb{T}^2 = \mathbb{R}^2_{\phi,\tau}/2\pi\mathbb{Z}^2$:

$$\begin{cases} \dot{\phi} = \frac{1}{\omega}(-\sin\phi + B + A\cos\tau) \\ \dot{\tau} = 1. \end{cases} \tag{1}$$

Physical problems of the Josephson junction led to studying the rotation number of system (1) as a function of the parameters (B, A) with fixed ω and to the problem on the geometric description of the *phase-lock areas:* the level sets of the rotation number function ρ with non-empty interiors. The rotation number has interpretation as average voltage over a long time interval (up to known constant) at the Josephson junction. In our case the phase-lock areas exist only for integer rotation numbers (quantization effect [2]). Each phase-lock area is a connected garland of a countable number of components going to infinity in the direction parallel to the A-axis [9]. Any two neighbor components are separated by one point, which is called *constriction* (provided it does not lie in the abscissa axis). On the complement to the phase-lock areas, which is an open set, the rotation number function ρ is an *analytic submersion* that induces its fibration by analytic curves [5].

Motivated by physical questions, we introduce a new function: the *Shapiro step function* $\mathcal{S}_{r,\omega}(A)$, whose modulus is equal to the length of the horizontal slice at level A of the phase-lock area with the rotation number r and given ω. It is an analytic function, whose zeros in the semi-axis $\{A > 0\}$ are the constrictions. Problem: *study analytic properties of the function $\mathcal{S}_{r,\omega}(A)$ and points $A_k(r,\omega)$ where $|\mathcal{S}_{r,\omega}(A_k(r,\omega))|$ is a local maximum.*

Dynamical systems (1) on torus can be complexified to Riccati equations and then transformed to a 3-parameter family of linear systems on the Riemann sphere with two irregular nonresonant singularities at 0 and at ∞: the so-called "systems of class J", which are equivalent to special double confluent Heun equations [2, 3, 4].

In the talk we give a survey of statements of problems and results of series of works [1–7, 9]. We discuss two very recent new results obtained in [1]. The first result states that in each phase-lock area with rotation number r all the constrictions lie on the same vertical line $\{B = r\omega\}$. It confirms the experimental fact discovered by S.I.Tertychnyi, D.A.Filimonov, V.A.Kleptsyn, I.V.Schurov in 2011, see [7]. The second result states that all the constrictions are "positive" (i.e., the corresponding germ of phase-lock area contains a germ of vertical line), solving a conjecture stated in [6]. Some of key arguments used in their proof are the following. *Systems of class J lie in a 4-dimensional family of linear systems that is the union of Jimbo isomonodromic families [8] given by solutions of a Painlevé 3 equation.* Systems of class J correspond to *order 1 poles of solutions* and thus, form a *local cross-section to these families.*

We present some open problems.

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A.A.Glutsyuk was supported by the Russian Science Foundation (RSF) (project no. 18-41-05003).

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APPROXIMATE QUASISYMMETRY

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Quasisymmetry is a spatial symmetry of first-order guiding-center motion that guarantees integrability. In this work, quasisymmetry is addressed in terms of approximate symmetries for arbitrary magnetic fields. Approximate quasisymmetry to leading order turns out to be the same as exact quasisymmetry. Phase-space symmetries of guiding-center motion introduce weak quasisymmetry. Nonetheless, magnetohydrostatics imposes quasisymmetry to leading order.

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ON SINGULARLY PERTURBED PARTLY DISSIPATIVE SYSTEMS OF EQUATIONS

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The system of equations

$$\begin{aligned} \frac{\partial u}{\partial t} + w(x)\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} &= \tilde{F}(u,v,x), \\ \frac{\partial v}{\partial t} + w(x)\frac{\partial v}{\partial x} &= \tilde{f}(u,v,x), \end{aligned} \tag{1}$$

is called a partially dissipative system of reaction-diffusion-transfer type. The word "partially" reflects the fact that the diffusion term is contained in only one of the two equations. In particular, such systems arise in mathematical models of chemical kinetics, where the sought functions $u(x,t)$ and $v(x,t)$ are the concentrations of reacting substances.

In the case of fast reactions, the reactive terms F and f contain large factors, namely the rate constants of fast reactions, due to which system (1) is reduced to the form characteristic of singularly perturbed systems of equations

$$\begin{aligned} \varepsilon^{\alpha}\Big(\frac{\partial u}{\partial t} + w(x)\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2}\Big) &= F(u,v,x), \\ \varepsilon^{\beta}\Big(\frac{\partial v}{\partial t} + w(x)\frac{\partial v}{\partial x}\Big) &= f(u,v,x), \end{aligned} \tag{2}$$

where $\varepsilon > 0$ is a small parameter, $\alpha > 0$, $\beta > 0$.

In recent years, for systems of the form (2), a number of problems have been considered, the solutions of which have boundary and internal transition layers [1-6]. The asymptotic expansions of these solutions have singularities due to the specifics of system (2).

In the talk an overview of the results obtained for systems of the form (2) is presented.

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STAR-TRIANGLE TRANSFORMATION OF THE POTTS MODEL PARTITION FUNCTION AS A SOLUTION FOR THE TETRAHEDRON EQUATION AND RELATED COMBINATORIAL TOPICS

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he talk is devoted to several well-known functional relations in the theory of polynomial graph invariants, as well as some new interpretations. The identification of Tutte polynomials with partition functions of the Potts-type models seems to be an ideal possibility to apply the methods from the theory of integrable models of statistical physics to the combinatorics of graphs and vice versa.

I will present at least one proof of the fact that the parameter transformation, defining the invariance of the n=2 Potts model partition function under the star-triangle transformation, gives an orthogonal solution for the local Yang-Baxter equation and for the tetrahedron equation. Using the Biggs duality on the space of Potts model partition functions I will present several results about the connection between the chromatic and flow polynomials and as a consequence obtain shifting order formulas on the space of Potts model partition functions.

The talk is based on the recent joint work with A.Kazakov and D.Talalaev.

SOLVABLE SYSTEMS OF TWO COUPLED FIRST-ORDER ODES WITH HOMOGENEOUS CUBIC POLYNOMIAL RIGHT-HAND SIDE

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The solution $x_n(t)$, $n = 1, 2$, of the *initial-values* problem is reported of the *autonomous* system of 2 coupled first-order ODEs with *homogeneous cubic polynomial* right-hand sides,

$$\dot{x}_n = c_{n1}(x_1)^3 + c_{n2}(x_1)^2 x_2 + c_{n3} x_1 (x_2)^2 + c_{n4}(x_2)^3 \ , \quad n = 1, 2 \ ,$$

when the 8 (time-independent) coefficients $c_{n\ell}$ are appropriately defined in terms of 7 *arbitrary* parameters, which then also identify the solution of this model. The inversion of these relations is also investigated, namely how to obtain, in terms of the 8 coefficients $c_{n\ell}$, the 7 parameters characterizing the solution of this model; and 2 *constraints* are *explicitly* identified which, if satisfied by the 8 parameters $c_{n\ell}$, guarantee the *solvability by algebraic operations* of this dynamical system. It is also identified a related, *appropriately modified*, class of (generally *complex*) systems, reading

$$\dot{\tilde{x}}_n = \mathbf{i}\omega\tilde{x}_n + c_{n1}(\tilde{x}_1)^3 + c_{n2}(\tilde{x}_1)^2 \tilde{x}_2 + c_{n3}\tilde{x}_1(\tilde{x}_2)^2 + c_{n4}(\tilde{x}_2)^3 \ , \quad n = 1, 2 \ ,$$

with $\mathbf{i}\omega$ an *arbitrary imaginary* parameter, which feature the remarkable property to be *isochronous*, namely their *generic* solutions are—as functions of *real time*—*completely periodic* with a period which is, for each of these models, a *fixed integer multiple* of the basic period $T = 2\pi/|\omega|$.

DISCRETIZATION OF THE NLS-TYPE HIERARCHY

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The non-linear Schrödinger (NLS) equation is one of the fundamental and widely studied equations in mathematical physics with broad application in physical problems, such as the non-linear optics and ocean waves. In this talk, we will present some recent results on the semi-discrete matrix NLS-like (DNLS) hierarchy. More precisely, we will construct the Lax pairs for the whole hierarchy via the dressing method and derive the corresponding soliton solutions via suitable choices of Darboux transformations.

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UNIVARIABLE FRACTAL INTERPOLATION FUNCTIONS

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We show how to construct a family of (discrete) dynamic systems, namely iterated function systems, whose graph is the attractor, a fractal set, of some continuous function which interpolates a given set of data. In particular, fractal interpolation functions which are widely presented in the literature can be obtained as particular cases of our construction.

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SINGULAR MEASURES AND INFORMATION CAPACITY OF TURBULENCE CASCADES

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How weak is the weak turbulence? Here we analyze turbulence of weakly interacting waves using the tools of information theory. It offers a unique perspective for comparing thermal equilibrium and turbulence. The mutual information between resonant modes in a finite box is shown to be stationary and small in thermal equilibrium, yet to grow with time in weak turbulence. We trace this growth to the concentration of probability on the resonance surfaces, which can go all the way to a singular measure. The surprising conclusion is that no matter how small is the nonlinearity and how close to Gaussian is the statistics of any single amplitude, a stationary phase-space measure is far from Gaussian, as manifested by a large relative entropy. This is a rare piece of good news for turbulence modeling: the resolved scales carry significant information about the unresolved scales. The mutual information between large and small scales is the information capacity of turbulent cascade, setting the limit on the representation of subgrid scales in turbulence modeling.

NONLINEAR DYNAMICS AND MECHANICS OF FLUIDS IN THE POROUS MEDIA

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The paper is devoted to the systems of nonlinear equations possessing applied significance in mathematical physics particularly in gas and fluid dynamics (Navier–Stokes equations), in physical kinetics (Boltzmann and Smoluchowski equations, Maxwell–Vlasov equations for media with discontinuities of parameters) and in phase transition models. The nonlinear operators in above equations are not continuous in Banach spaces specific for these conservation laws. There are discussed general mathematical structures, connected with approximate methods convergence. The existence and uniqueness theorems for global solutions of the Cauchy problem for quasilinear and semilinear systems are proved. The problem of passage to the limit via small parameters for singularly perturbed problems are considered. The problems of computations in above models are discussed too.

We consider the problem of connectedness in the metric space which model is so called porous medium. The main question is the description of measure for quantity of global connections between two points in the space provided the local distribution of connections in micro-structure is given. A porous medium is a network of inter-grain channels formed by internally connected intermediate spaces between particles. Natural examples of above problems are investigated in description of global structures produced by connected pores in matrix of oil-containing sands and similar problems arise in research of materials of nuclear reactors under influence of neutron flow.Those problems are close to the description of structures in multidimensional billiard game. The description of global structures in the above examples connected with solutions of Smoluchowskii kinetic Equation [1], which is directly leads to the non-local Hopf Equation [2] for density distribution function of global conductivity paths in structure of porous medium.

The mathematical models of physical systems, consisting of statistically plenty of particles (rare gases, dispersive systems, plasma, systems with phase transition surfaces) and continuous medium mechanics models are based on fundamental relations of the balance which general name is *conservation laws.*

Their applications are well-known, particularly, in connection with gas dynamics equations and hydrodynamics, physical kinetics equations by Boltzmann and Smoluchowski, plasma theory, models of crystal growth etc. [1].

The extension of the concept of a solution (*functional solutions*) [1] makes it possible to justify the convergence of approximate methods in presence of an a'priori estimate of approximations in $L_1^{loc}(Q,\nu)$, which is uniform in the parameter provided nonlinear operators are not continuous in this space.

Theorem. *Let the approximate (singularly perturbed) method has properties of uniform weak approximation and uniform weak stability in Tikhonov topology. Then we can point out uniqueness class of functional solutions and we can construct in it correctness class of the Cauchy problem $\mathcal{U}$ such that on some directed set of the parameters approximations converge to the points in the set $\mathcal{U}$ for given set of initial data.*

We apply this Theorem to the for singularly perturbed problem for systems with small parameters (viscosity method)

$$\frac{\partial u^{(i)}(x,t)}{\partial t}+\sum_{j=1}^{n}\frac{\partial f_j^{(i)}(u,x,t)}{\partial x_j}=\alpha_i\sum_{j=1}^{n}\frac{\partial^2 u^{(i)}}{\partial x_j^2},\quad u|_{t=0}=u_0,$$

$$t>0,\quad x\in\mathbb{R}_n,\quad 1\le i\le m,\quad \alpha=(\alpha_1,\alpha_2,\ldots,\alpha_m),\quad \alpha_i\ge 0.$$

In this case we obtain description of correctness classes consisting of functional solutions.

The exact solutions of Navier–Stokes Equations and MHD for incompressible fluids in porous space are presented.

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FOKAS-LENELLS EQUATIONS ON HERMITIAN SYMMETRIC SPACES

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We formulate multi-component integrable generalizations of the Fokas-Lenells equation [1,2] which are associated with each irreducible Hermitian symmetric space. We provide a description of the underlying structures associated to the integrability, such as Lax formulation and the bi-Hamiltonian formulation of the equations. Two reductions are considered as well one of which leads to a nonlocal integrable model. Two examples with symmetric spaces of types A.III and BD.I are presented in details.

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RECURSION OPERATORS AND HIERARCHIES OF MKDV EQUATIONS RELATED TO THE KAC-MOODY ALGEBRAS $A_5^{(1)}$ AND $A_5^{(2)}$

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This work is an extension of our results in [1, 2]. Here we construct the two nonequivalent gradings in the algebra $A_5 \simeq sl(6)$. The first one is the standard one obtained with the Coxeter automorphism $C_1 = S_{\alpha_1}S_{\alpha_3}S_{\alpha_5}S_{\alpha_2}S_{\alpha_4}$ using its dihedral realization. In the second one we use $C_2 = C_1R$ where R is the external automorphism of A_5. For each of these gradings we constructed the basis in the corresponding linear subspaces $\mathfrak{g}^{(k)}$, the orbits of the Coxeter automorphisms and the related Lax pairs generating the corresponding mKdV hierarchies. We found compact expressions for each of the hierarchies in terms of the recursion operators. At the end we wrote explicitly the first nontrivial mKdV equations and their Hamiltonians. We also derived the completeness relations for the 'squared solutions' of the Lax operator and use them to: i) prove that the inverse scattering method is a generalized Fourier transform, and ii) to obtain explicit expressions for the action-angle variables of the MKdV equations.

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FOURIER TRANSFORM METHOD FOR SOME TYPES OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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In [1], based on ideas from statistical hydrodynamics related to the Hopf equation, a method is presented that makes it possible to apply the Fourier transform to solve the Cauchy problem for a wide class of equations of the form

$$\partial_t u(t,x) + \sum_{|\alpha|\le m} \varepsilon_\alpha a_\alpha(t)\partial_x^\alpha u(t,x) = f(t,x), \qquad (1)$$

where the factors ε_α coincide with either 1 or one of the spatial variables. In the same paper, the possibility of a slightly different approach is indicated, when the application of the Fourier transform reduces the original problem to the Cauchy problem for first-order partial differential equations. In this paper, based on this approach, we propose a method for analyzing the Cauchy problem for evolutionary partial differential equations with power nonlinearities. The method is based on the Fourier transform, which allows rewriting the original equation as an integro-differential one, in which the integration is carried out not over the time variable, over which the equation contains the derivative, but over the spatial one. The space of coefficients is defined by theorems of the Paley-Wiener-Schwarz type on the Fourier transform so that the known functions under the integral in the integro-differential equation are compactly supported - we use this to derive some a priori estimates.

We now move to an exact formulation. We introduce the spacess: $C^{l,A}_{F^m,T}(R^n) := \{\Phi|_{[0,T]\times R^n} | (\Phi(\cdot,x) \in C^l(0,T) \forall x \in R^n) \wedge (\forall t \in [0,T]$ $\Phi(t,\cdot) : C^n \to C)$ is an entire function: $(\Phi(t,x) = 0\ \forall (t,x) \in [0,T] \times R^n) \wedge (\exists\, c = c(\Phi), r \in R : |\Phi(z)| < c(1+||z||_{C^n})^{-m} e^{r|z|}\ \forall (t,z) \in [0,T]\times R^n)\}$; $l, m \in N \cup \{0\}$ - the space of coefficients; we observe that, if $a_\alpha \in C^{0,A}_{F^1,T}(R^n)$, then from the Paley-Wiener theorem $\sup a^{\sqcup}_\alpha(t,\cdot) \subset\subset R^n$, where $a^{\sqcup}_\alpha(t,\cdot)$ is differentiable and $\partial_{x_j} a^{\sqcup}_\alpha(t,\cdot) \in L_2(R^n)\ \forall j$; $C^A_L(R^n) = \{\Phi|_{R^n} : (\Phi : C^n \to C$ is an entire function: $((Im\Phi(x) = 0\ \forall x \in$

$R^n)\wedge(\int_{R^n}|\Phi(x)|^2dx<\infty)\wedge(\exists c,r\in R:|\Phi(z)|<ce^{r|Imz|_{R^n}}\ \forall z\in C^n))\}$ is the space of initial data. In what follows $C^{l,A}_{F,T}(R^n)$ means that $C^{l,A}_{F^0,T}(R^n), H^m(R^n)$ are Sobolev spaces.

The main result is the following Theroem.

Theorem. Assume that for some T_1 in the problem

$$\begin{gathered}\partial_t u(t,x)+\sum_{k=0}^{k_0}\sum_{|\alpha^1|\le M,\Box,|\alpha^l|\le M}\varepsilon_{\alpha^1\Box\alpha^l}a^k_{\alpha^1,\Box,\alpha^l}(t,x)u^k\partial_x^{\alpha^1}u(t,x)\cdot\\ \Box\cdot\partial_x^{\alpha^l}u(t,x)=f(t,x);\\ u|_{t=0}=u_0(x)\in C^A_L(R^n)\end{gathered}\tag{2}$$

$\varepsilon_{\alpha^1\Box\alpha^l}=1$ for even $\sum_{j=1}^l|\alpha^j|$ and in the opposite case $\varepsilon_{\alpha^1\Box\alpha^l}$ is equal to one of the spatial variables x_i; $f\in C^{0,A}_{F,T_1}(R^n)$; $a^{lk}_\varepsilon\in C^{0,A}_{F,T_1}(R^n)$ for even $\sum_{j=1}^l|\alpha^i|$ and $a^{lk}_\alpha\in C^{0,A}_{F^1,T_1}(R^n)$ in the opposite case; let m be an arbitrary natural number with $m>M+n2^{-1}$. Then $\exists C=C(a^k_{\alpha^1\ldots\alpha^l},f(t,x),n,m,M)>0,\ \exists T\in(0,T_1]$ such that for initial conditions satisfying $\int|u_0(x)|^2dx<C$, there exists $u\in L_1([0,T];\ H^m(R^n))$ such that $u(\cdot,x)\in C^1(0,T)\ \forall x\in R^n$, u is a solution of problem (2).

Note that problem (2) is not only of purely theoretical interest, but also of applied – there is a large number of equations of the form (2) that describe real processes (see, for example, [2]).

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LOCAL SOLUTIONS OF SLOW-FAST DELAY OPTOELECTRONIC MODEL

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We consider the delay optoelectronic oscillator model (see [1] and references)

$$\varepsilon \frac{dx}{d\zeta} = y - x + \beta\big[\cos^2(x(\zeta-\nu)+\phi) - \cos^2\phi\big], \qquad \frac{dy}{d\zeta} = -x.$$

Substituting $\zeta = \nu t$ and transformating we obtain the second-order equation

$$\frac{\varepsilon}{\nu}\ddot{x} + \dot{x} + \nu x = b_1\dot{x}(t-1) + 2b_2 x(t-1)\dot{x}(t-1) + 3b_3 x^2(t-1)\dot{x}(t-1) + \ldots,$$

$$\text{where} \quad b_1 = -\beta\sin(2\phi), \quad b_2 = -\beta\cos(2\phi), \quad b_3 = 2\beta\sin(2\phi)/3.$$

We consider some critical cases in zero solution stability problem. In the vicinity of the zero steady state other solutions can be represented as a series by degrees of a small parameter ε.

To define the main part of solutions we construct special non-linear boundary value problems with periodic boundary conditions as normal forms without small parameter. Non-local dynamics of BVPs specifies local behavior of origin equation solutions.

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NEIMARK–SACKER BIFURCATION AND STABILITY ANALYSIS FOR FAMILY OF MAPS MODELLING DELAYED LOGISTIC EQUATION

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Difference approximations of the logistic equation with delay or the Hutchinson equation

$$\frac{du}{dt} = r[1 - u(t-1)]u. \tag{1}$$

are studied. This equation is widely used in problems of mathematical ecology and belongs to the fundamental models of population dynamics [1,2]. The nonnegative function $u(t)$ models the normalized population density, and the positive parameter r characterizes the rate of its growth. Numerical analysis of equation (1) implies the construction of families of difference equations, the accuracy of calculating the solution with the help of which depends, first of all, on the chosen approximation step. The smaller the step is chosen, the more accurate the calculations are. At the same time, the mappings under study can in themselves serve as models of population dynamics. Therefore, their study and comparison of dynamic properties with the original continuous equation is of considerable interest.

Variants of difference approximations of the Hutchinson equation were analyzed in papers [3–5]. To obtain the corresponding mappings, in [3,4] the derivative with respect to t was replaced by the difference forward or backward, and in [5] the mapping was constructed on the basis of difference approximations of an integral equation equivalent to the equation (1). We consider a mapping constructed on the basis of approximating the time derivative using the central difference.

We fix an arbitrarily natural k and assume that the time step is equal to $1/k$. If the time derivative in (1) is replaced by the central difference $\dfrac{u(t+1/k) - u(t-1/k)}{2/k}$, than we obtain the difference equation of order $k+1$

$$u_{n+1} = u_{n-1} + \frac{2r}{k}(1 - u_{n-k})u_n, \quad n \geqslant 0, \qquad (2)$$

where $t = n/k$, $n \in \mathbb{Z}$, and the value $u(n/k)$ is replaced by u_n.

One of the fundamentally important properties of the equation (1) is that it has an orbitally asymptotically stable cycle for $r > \pi/2$ (see [1,2]). This cycle bifurcates from the equilibrium state $u \equiv 1$ as a result of the Andronov – Hopf bifurcation. For a model mapping, this cycle corresponds to a stable invariant curve (see [3–6]).

We have proved that for critical values of the parameter r, at which in problem (2) the solution $u \equiv 1$ loses its stability, the characteristic polynomial of equation (2) linearized on this solution has two pairs of roots (without resonances) on the unit circle of the complex plane.

When constructing the normal form of equation (1) it turned out that for such an approximation this equation ceases to adequately model the dynamics of the Hutchinson equation. In particular, for the values of the parameter r corresponding to the loss of stability of a unit equilibrium state, a stable invariant curve does not branch off from it, to which a stable cycle (stable periodic oscillations) corresponds to the Hutchinson equation. This shows that when choosing a difference scheme that simulates the dynamics of the Hutchinson equation, it is not always possible to obtain a difference equation with the required properties.

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CONSTRUCTION OF CYCLES WITH A DIFFERENT NUMBER OF BURSTS FOR EACH OF THE COMPONENTS IN THE RING OF RELAY OSCILLATORS

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Let us consider a ring network of m oscillators with a unidirectional synaptic coupling

$$\begin{gathered} \dot u_j = \lambda \left[F(u_j(t-1)) + G(u_{j-1}(t-h))\ln(u_*/u_j)\right] u_j, \\ j = 1, \ldots, m, \quad u_0 \equiv u_m. \end{gathered} \tag{1}$$

Here $u_j > 0$ is a normalized neural membrane potentials, $\lambda \gg 1$ characterises the rate of electric processes in the system, $u_* = \exp(c\,\lambda)$, $c = \mathrm{const} \in \mathrm{R}$, the relay functions $F(u)$ and $G(u)$ have the form

$$F(u) \stackrel{\text{def}}{=} \begin{cases} 1, & 0 < u \le 1, \\ -a, & u > 1, \end{cases} \quad G(u) \stackrel{\text{def}}{=} \begin{cases} 0, & 0 < u \le 1, \\ b, & u > 1. \end{cases}$$

$a, b = \mathrm{const} > 0$.

Fix natural numbers $k_1, \ldots, k_m$. In this work we construct a periodic solutions such that j-th component has k_j high bursts after long enough segment with small values.

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PERIODIC RELAXATION SOLUTIONS OF CERTAIN GENERALIZATION OF LOGISTIC EQUATION WITH STATE-DEPENDENT DELAY

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Consider logistic equation with state-dependent delay:

$$\dot{N} = \lambda N \left[1 - N(t - h(\lambda) - f(N(t-T)))\right],$$

where λ is sufficiently large ($\lambda \gg 1$) and state depends on time in the past. This equation is a generalization of state-dependent delay equation considered in [1].

Under certain assumptions about functions $h(\lambda)$ and $f(N)$, the next theorem holds:

Theorem 1. *If $\lambda \gg 1$, then original equation has nonlocal relaxation periodic solution $N^*(t, \lambda)$. The initial condition of this solution belongs to the convex, bounded and closed set.*

Asymptotic properties of solution $N^*(t, \lambda)$ were also investigated. Namely, if $\lambda \gg 1$, then the period and the amplitude of this solution are asymptotically large, and its minimal value is asymptotically small.

We used the method of the big parameter [2] in order to establish these facts.

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GRASSMANN EXTENSIONS OF YANG–BAXTER MAPS

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In this talk we show that there are explicit Yang-Baxter maps with Darboux-Lax representation between Grassmann extensions of algebraic varieties. Motivated by some results on noncommutative extensions of Darboux transformations, we first derive a Darboux matrix associated with the Grassmann-extended derivative Nonlinear Schrödinger (DNLS) equation, and then we deduce novel endomorphisms of Grassmann varieties, which possess the Yang–Baxter property. In particular, we present ten-dimensional maps which can be restricted to eight-dimensional Yang-Baxter maps on invariant leaves, re- lated to the Grassmann-extended NLS and DNLS equations. We consider their vector generalisations.

Part of this talk is in joint collaboration with S. Konstantinou-Rizos and A. V. Mikhailov [1].

We discuss the integrability of such maps.

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QUANTUM DOUBLES AND QUANTUM VERTEX ALGEBRAS

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I plan to introduce some new classes of quantum algebras which can be hopefully useful for constructing certain deformations of matrix models. The role of the Reflection Equation algebra in their construction will be explained. The notion of quantum vertex algebra (Etingof, Kazhdan, Kac and others) will be discussed.

GENERALIZED INVARIANT MANIFOLDS FOR INTEGRABLE EQUATIONS AND THEIR APPLICATIONS

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In the talk a notion of the generalized invariant manifold for nonlinear integrable equation will be discussed. Recently in our works [1-6] it has been observed that this kind objects provide an effective tool for evaluating the Lax pairs and recursion operators.

The approach developed in [1-6] explains the essence of the Lax pair phenomenon. In fact, the Lax pair in $1+1$ dimension is naturally (internally) derived from the nonlinear equation under consideration. First we find the linearization (Frechet derivative) of the nonlinear equation. The linearized equation obviously includes the dynamical variables of the original equation as well, which are already considered as functional parameters. Now we find an ordinary differential equation, also depending on the dynamical variables of the original equation, compatible with the linearized equation. We call this ordinary differential equation a generalized invariant manifold. There exist many such manifolds, among them there are also nonlinear ones. In order to evaluate the generalized invariant manifolds we use the consistency with the linearized equation which allows to derive a system of differential (difference) equations that is highly overdetermined due to the presence of the independent parameters – dynamical variables of the original nonlinear equation. In all of the examples studied (KdV, mKdV, Kaup-Kupershmidt equation, Krichever-Novikov equation, Volterra type lattices from Yamilov list, two equations of KdV type found by Svinolupov and Sokolov, Garifullin-Mikhailov-Yamilov non-autonomous lattice, sine-Gordon equation and several hyperbolic type equations, etc) the corresponding overdetermined systems are effectively solved and the desired non-trivial manifolds are found. Trivial generalized invariant manifolds are constructed quite elementarily by using the classical symmetries. A manifold that is consistent with the linearized equation if and only if the original nonlinear equation is satisfied is called non-trivial. In essence, this requirement means that a pair consisting of a linearized equation and a generalized invariant manifold defines a Lax pair. It is curious that usual Lax pairs do not belong to this class, but they can be derived from this class by suitable transformations. Note that new Lax pairs are interesting in themselves. For example, an invariant manifold corresponding to a nontrivial linear pair of the lowest order is very easily transformed into the recursion operator.

Since the invariant manifold deals with two sets of dynamical variables defined by the equation in question and the linearized equation, it has two orders. Sometimes the problem of determining orders causes problems. The corresponding relationship between these two orders for the case of equations of the KdV type was established in [7].

In [6] it was shown with the example of the Volterra lattice that a nonlinear Lax pair can be used for constructing particular solutions of the original nonlinear equation. To this end we first find a non-trivial invariant manifold depending on two constant parameters. Then we assume that ordinary difference equation defining the generalized invariant manifold has a solution polynomially depending on one of the parameters. The assumption is rather severe, it produces some ordinary difference and differential equations, providing a separation of the variables. Application of the method is illustrated by examples.

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CHAOTIC DYNAMICS IN A PLANAR MODEL OF GRAPHENE

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Using a two dimensional atomic Hamiltonian model of graphene [1], we investigate the material's chaotic dynamics through numerical simulations implementing symplectic integration techniques [2]. We study periodic graphene sheets (modelling the bulk behaviour) as well as finite width graphene nanoribbons (GNRs) across a range of temperatures, considering both the usual ^{12}C isotopes as well as the effect of doping with ^{13}C. By computing the maximal Lyapunov Exponent (MLE) (see e.g. [3]) we quantify the chaoticity of these structures, also comparing the stability of armchair and zigzag edge GNRs [4]. The resultant Lyapunov time, i.e. the inverse of the MLE, is compared with the characteristic time scales of the fastest normal modes, providing context to the rate of chaotisation in graphene. Several simplifying modifications of the Hamiltonian are also briefly considered, demonstrating that chaos is inherent in the two dimensional geometry, with simple harmonic interatomic couplings yielding positive MLE values.

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NONCOMMUTATIVE KEPLER DYNAMICS: SYMMETRY GROUPS AND BI-HAMILTONIAN STRUCTURES

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Integrals of motion are constructed from noncommutative (NC) Kepler dynamics, generating $SO(3)$, $SO(4)$, and $SO(1,3)$ dynamical symmetry groups. The Hamiltonian vector field is derived in action-angle coordinates, and the existence of a hierarchy of bi-Hamiltonian structures is highlighted. Then, a family of Nijenhuis recursion operators is computed and discussed.

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YANG–BAXTER MAPS ASSOCIATED WITH DARBOUX TRANSFORMATIONS, LIE GROUPS, AND LINEAR APPROXIMATIONS OF REFACTORISATION PROBLEMS

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Many (but not all) results of my talk are presented in the joint work with V.M. Buchstaber, S. Konstantinou-Rizos, and M.M. Preobrazhenskaia [1].

Yang–Baxter maps (YB maps) are set-theoretical solutions to the quantum Yang–Baxter equation. Relations of YB maps with integrable systems (including integrable PDEs and lattice equations) and with various algebraic structures are very active areas of research.

For a set $X = \Omega \times V$, where V is a vector space and Ω is regarded as a space of parameters, a linear parametric YB map is a YB map

$$Y\colon X \times X \to X \times X$$

such that Y is linear with respect to V and one has $\pi Y = \pi$ for the projection

$$\pi\colon X \times X = (\Omega \times V) \times (\Omega \times V) \to \Omega \times \Omega.$$

These conditions are equivalent to certain nonlinear algebraic relations for the components of Y. Such a map Y may be nonlinear with respect to parameters from Ω.

In my talk, I will present general results on such maps, including clarification of the structure of the algebraic relations that define them and several transformations which allow one to obtain new such maps from known ones. Also, methods for constructing such maps will be described.

In particular, developing an idea from [Konstantinou-Rizos S. and Mikhailov A. V. 2013 J. Phys. A: Math. Theor. 46 425201], I plan to demonstrate how to obtain linear parametric YB maps from nonlinear Darboux transformations of some Lax operators, using linear approximations of matrix refactorisation problems corresponding to Darboux matrices. Also, I will present a wide class of new linear parametric

YB maps (with nonlinear dependence on parameters) associated with Lie groups.

I acknowledge the support by the Russian Science Foundation (grant No. 20-71-10110).

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RELAXATION MODES IN THE RING OF OSCILLATORS WITH DELAYED FEEDBACK

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Consider a mathematical model of a ring of the simplest generators [1] with the nonlinear delayed feedback

$$\dot{u}_j + u_j = \lambda F(u_j(t-T)) + \gamma(u_{j-1} - 2u_j + u_{j+1}), \quad (j = 1, \ldots, N), \\ u_0 \equiv u_N, \quad u_{N+1} \equiv u_1. \tag{1}$$

Here u_j $(j = 1, \ldots, N)$ are scalar functions, $N = 2k+1$ $(k = 1, 2, \ldots)$, non-zero coupling parameter γ is some constant greater than $-\frac{1}{4}$ (at these values of the parameter γ system (1) is dissipative), $F(\cdot)$ is a smooth nonlinear compactly supported function (it is equal to zero, when absolute value of function argument is greater than 1), positive parameter λ is sufficiently large ($\lambda \gg 1$).

We study the nonlocal dynamics of model (1) in the phase space $C([-T, 0]; \mathbb{R}^N)$.

In the case $\gamma > 0$ we find the asymptotics of the homogeneous relaxation cycle and show that all solutions with initial conditions from some region of the phase space after some time have the same leading part of the asymptotics.

In the case $-\frac{1}{4} < \gamma < 0$ we calculate asymptotics of all solutions of system (1) with initial conditions from some region of the phase space. Based on asymptotics we construct auxiliary two-dimensional map. This map determines dynamics of initial system: rough fixed point (cycle) of this map corresponds to relaxation cycle of initial system (1). We prove that this map has at least N cycles, so initial system has at least N coexisting relaxation cycles.

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THE DYNAMICS OF SINGULAR PERTURBED SYSTEM OF TWO DELAY DIFFERENTIAL EQUATIONS

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Consider system of two delay differential equations

$$\begin{aligned} &\gamma^{-1}\dot{x} + x = x(t-T)(a + d_1 y + d_2 y^2),\\ &\dot{y} = by + cx^2. \end{aligned} \tag{1}$$

This problem is somewhat simplified model of FDML-laser [1]. Let study the dynamics of (1) in small neighbourhood of equilibrium in the phase space $C_{[-T;0]} \times \mathbb{R}$.

Main assumption is that the value of γT is large enough. Thus, system (1) is singularly perturbed. This can be done in one of three main cases:
1) γ is large;
2) T is large;
3) both γ and T are large.

Denote $\varepsilon = (\gamma T)^{-1} \ll 1$ and make time substitution $t \to tT$:

$$\begin{aligned} &\varepsilon\dot{x} + x = x(t-1)(a + d_1 y + d_2 y^2),\\ &\varepsilon^{\mu}\dot{y} = by + cx^2, \quad \mu \geq 0. \end{aligned} \tag{2}$$

Let $b < 0$, so the equilibrium state loses stability if $|a| = 1$. In this case the asymptotically large number of roots of the corresponding characteristic equation (spectrum points) lies arbitrarily close to the imaginary axis. Thus, critical cases have infinite dimension.

Let a is close to ± 1: $a = \pm(1 + \varepsilon^p a_1)$, $p > 0$.

In the critical cases special nonlinear equations are constructed - quasinormal forms - which do not depend on a small parameter or depend on it regularly. Solutions of quasinormal forms determine the main parts of the asymptotic expansion of solutions (1).

The quasinormal forms are nonlinear parabolic boundary problem. Its' exact form is strongly depend on relationship between small parameters ε, ε^{μ} and ε^p.

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INFINITE-DIMENSIONAL TURING BIFURCATION IN CHAINS OF CONNECTED VAN DER POL SYSTEMS

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Consider the local dynamics in the neighbourhood of the equilibrium state of ring of coupled van der Pol equations. The main assumption, which opens the way to the application of asymptotic methods, is that the number of N elements in the chain is sufficiently large. The transition from a discrete system to a system with a continuous spatial variable is naturally carried out. It is shown that critical cases in the stability problem have infinite dimension. As the main results, special nonlinear boundary value problems of a parabolic type are constructed, which play the role of a first approximation equation. The nonlocal dynamics of these boundary value problems describes the local behavior of the solutions of the original system.

INTEGRABLE TWO-COMPONENT SYSTEMS OF DIFFERENCE EQUATIONS

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We will present two lists of two-component systems of integrable difference equations defined on the edges of the $\mathbb{Z}^2$ graph. The integrability of these systems is manifested by their Lax formulation which is a consequence of the multi-dimensional compatibility of these systems. Imposing constraints consistent with the systems of difference equations, we recover known integrable quad-equations including the discrete version of the Krichever-Novikov equation. The systems of difference equations give us in turn quadrirational Yang-Baxter maps.

STABILITY OF PIECEWISE SMOOTH SOLUTIONS OF A DISTRIBUTED DYNAMICAL SYSTEM

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Consider the spatially distributed equation

$$\dot{\xi} = \xi - \beta(\xi^2 - M(\xi^2)) - (1-\beta)(\xi^3 - M(\xi^3)),$$

with boundary conditions

$$\xi(t, x+1) = \xi(t, x),$$
$$M(\xi) = 0,$$

where $\beta \in [0, 1]$, $\xi = \xi(t, x)$ is a step function of the variable x for each $t \geq 0$, $M(\xi) = \int\limits_0^1 \xi(t, x)dx$.

In some cases, this boundary value problem is the simplified quasi-normal form of a model of optoelectronic oscillator (see [1]).

In this research we proved that in this boundary value problem there exists a one-parameter family of solutions depending on the parameter $\alpha \in (0, 1)$ in the form of step functions

$$\xi(t, x) = \begin{cases} 0, & x = 0 \\ a(\alpha), & 0 < x < \alpha \\ 0, & x = \alpha \\ b(\alpha), & \alpha < x < 1 \\ 0, & x = 1 \end{cases}.$$

We proved that if $\beta = 1$ then for any $\alpha \in (0, 1)$ these solutions are unstable. If $\beta = 0$ then these solutions are stable for $\frac{1}{3} < \alpha < \frac{2}{3}$.

In the other cases, for any $\beta \in (0, 1)$ there exist an interval (α_1, α_2) in which these solutions are stable.

This research was supported by the Russian Foundation for Basic Research (grant No. 18-29-10055).

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CLUSTER MAPS ASSOCIATED WITH THE DISCRETE KDV EQUATION

Theodoros Kouloukas

Nonlinear recurrences arise from cluster algebras with periodicity. In this talk we will study a class of nonlinear recurrences from cluster mutation-periodic quivers obtained as reductions of the discrete Hirota equation, which are related to travelling wave solutions of the lattice KdV equation. We will demonstrate the integrability in the Liouville sense of the associated birational maps using the properties of the underlying cluster algebra structure.

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QUADRATIC CONSERVATION LAWS FOR EQUATIONS OF MATHEMATICAL PHYSICS

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Linear systems of differential equations in a Hilbert space are considered that admit a positive-definite quadratic form as a first integral. The following three closely related questions are the focus of interest in this paper: the existence of other quadratic integrals, the Hamiltonian property of a linear system, and the complete integrability of such a system. For non-degenerate linear systems in a finite-dimensional space essentially exhaustive answers to all these questions are known. Results of a general nature are applied to linear evolution equations of mathematical physics: the wave equation, the Liouville equation, and the Maxwell and Schrödinger equations.

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HIGHER-ORDER ANALOGUES OF THE PAINLEVé EQUATIONS ASSOCIATED WITH SAWADA-KOTERA AND KUPERSHMIDT HIERARCHIES AND THEIR PROPERTIES

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The self-similar reductions of the Sawada-Kotera and the Kupershmidt equations are studied. Results of Painlevé test for these equations are given. Lax pairs for soling the Cauchy problems to these nonlinear ordinary differential equations are found. Special solutions of the Sawada-Kotera and the Kupershmidt expressed via the first Painlevé equation are presented. Exact solutions of the Sawada-Kotera and the Kupershmidt equations by means of general solution for the first member of K_2 hierarchy are given. Special polynomials for expressions of rational solutions for considered equations are introduced. The differential-difference equations for finding special polynomials corresponding to the Sawada-Kotera and the Kupershmidt equations are found. Nonlinear differential equations of sixth order for special polynomials associated with the Sawada-Kotera and the Kupershmidt equations are obtained. Lax pairs for nonlinear differential equations with special polynomials are presented. Rational solutions of the self-similar reductions for the Sawada-Kotera and the Kupershmidt equation are given.

A LEAN WARNING MODEL FOR RECOGNITION OF PANDEMIC SCALE DANGER OF VIRUS INFECTIONS

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A simple two parametric SIS-model of infection spreading has been developed and solved. It describes two processes, namely infection and recovery/deaths. Its solution has a quasi-logistic functional form and includes a naturally arisen constant combination perceived as an infection index. When tested on SARS Cov-2 pandemic as well as influenza epidemic data, the model have shown its promising potential for forecasts. A full-featured SIR-model affined with the SIS-model has been built and solved.

A lot of epidemiological models having been built in the past, concentrate typically on accuracy in reproduction of approximated pandemic data, developing both analytical and numerical methods as well as on attempts to build universal or lean (with focus on finally expected pandemic results rather than process behaviour) approaches to grasp the general or final picture [1-3]. In the present work the attempt was undertaken to develop a light model able to recognise potential pandemics on the course of expansion initial phase, so that almost everyone could use it for tests on publicly available data. Considering a number of infected $n(t)$, we assume N to represent a community population with free contacts among individuals within. If $N - n(t)$ is a number of susceptibles, the differential equation of the infection and recovery/death process reads:

$$n_t = \alpha n(N - n) - \beta n. \tag{1}$$

(where $\alpha > 0$ is contact and $\beta \geq 0$ recovery/death rate). A Riccati-type Eq. (1) has the solution

$$n(t) = \frac{M(\alpha N - \beta)}{M\alpha + (\alpha N - \beta - M\alpha)\exp[-(\alpha N - \beta)t]}. \tag{2}$$

The results of a least square method approximation according to Eq. (2) are presented in Fig. 1. While compared with that for influenza epidemic data, infection index $h = \alpha N - \beta$ has been found to have a threshold $h = 0.13$ above which the course of infection pretends to be characterised as a pandemic and below that as epidemic.

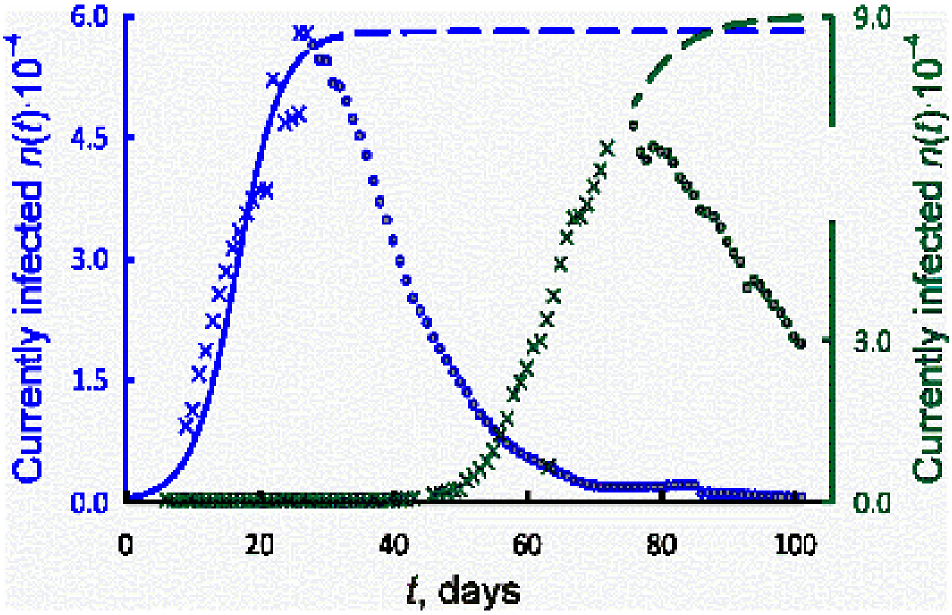

Fig. 1: Dependence of the number of infected on time in China (blue, from 22-Jan till 19-Feb-2020, $M = 547, \alpha = 5.1007 \cdot 10^{-6}, \beta = 7162.76, h = 0.2969, R^2 = 0.969$) and Germany (green, 28-Jan till 08-Apr-2020, $M = 4, \alpha = 1.8568 \cdot 10^{-6}, \beta = 154.22, h = 0.16688, R^2 = 0.995$). Data source [4]. Crosses and circles: processed data; points: unused data (pandemic decrease); solid – approximating curves, dashed – segments of those for the time periods corresponding to the decreasing pandemic data.

A sister, full-featured SIR-model has been built that includes the same basic ideas as the SIS-model but more accurately treats the number of susceptibles $N(t)$:

$$n_t = \alpha n\Big(N(0) - \beta \int_0^t n(\tau)d\tau - n\Big) - \beta n.$$

Through a new function representing n_t/n, it can be transformed into a 2^{nd} order ordinary differential equation whose general solution has the form (n_p is the infectives peak value):

$$t = t_0 + \int \frac{ndn}{\left\{ \exp\left[\frac{\alpha(n-n_p)-\beta(1-\log\beta)}{\beta} - \frac{1}{\beta}W\left(-\frac{1}{\beta}e^{\frac{\alpha(n-n_p)-\beta(1-\log\beta)}{\beta}}\right)\right] - \beta\right\}},$$

where $W(x)$ is the Lambert function that is real-valued for $x \geqslant -1/e$ with two branches.

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CUBIC-QUINTIC-SEPTIC OPTICAL SOLITONS IN THE OPTICAL FIBER BRAGG GRATINGS OF THE NONLINEAR DIFFERENTIAL EQUATION

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Travelling wave reduction of the system of two equations describing the propagation of nonlinear waves in the optical fiber Bragg gratings is considered. The compatibility conditions for the overdetermined system of equations are found. Solitary wave solutions in optical fiber gratings are demonstrated.

In this work we consider the system of two equations in the following form:

$$\begin{gathered} iu_t + ic_1u_x + c_2u_{xx} + ic_3u_{xxx} + c_4u_{xxxx} + ic_5u_{xxxxx} + c_6u_{xxxxxx} + \\ +(c_7|u|^2 + c_8|v|^2)u + (c_9|u|^4 + c_{10}|u|^2|v|^2 + c_{11}|v|^4)u + \\ +(c_{12}|u|^6 + 3c_{13}|u|^4|v|^2 + 3c_{14}|u|^2|v|^4 + c_{15}|v|^6)u + sv = 0, \\ iv_t + id_1v_x + d_2v_{xx} + id_3v_{xxx} + d_4v_{xxxx} + id_5v_{xxxxx} + d_6v_{xxxxxx} + \\ +(d_7|u|^2 + d_8|v|^2)v + (d_9|u|^4 + d_{10}|u|^2|v|^2 + d_{11}|v|^4)v + \\ +(d_{12}|u|^6 + 3d_{13}|u|^4|v|^2 + 3d_{14}|u|^2|v|^4 + d_{15}|v|^6)v + ru = 0, \end{gathered} \tag{1}$$

where ru and sv are additional terms responsible for the reflective properties of the waves.

The system (1) corresponds to the sixth order equation with cubic-quintic-septic law nonlinearity considered in [1, 2, 3]. Eq.(1) describes pulse propagation in the optical fiber Bragg gratings for two waves.

Using traveling wave reductions

$$u(x,t) = y_1(z)e^{i(kx-kx_0-\omega t)}, \quad v(x,t) = y_2(z)e^{i(kx-kx_0-\omega t)}, \quad z = x - C_0t, \tag{2}$$

we obtain the overdetermined system for $y_1(z)$ and $y_2(z)$, consisting of four equations.

Getting the compatibility conditions of the obtained system we obtain a new system:

$$\begin{gathered} y_{1,zzzzzz}c_6+(15k^2c_6+c_4)y_{1,zzzz}+(75k^4c_6+6k^2c_4+c_2)y_{1,zz}+y_1^7c_{12}+ \\ +3y_1^5y_2^2c_{13}+y_1^5c_9+3y_1^3y_2^4c_{14}+y_1^3y_2^2c_{10}+y_1^3c_7+61y_1k^6c_6+y_1y_2^6c_{15}+ \\ +5y_1k_4^c+y_1y_2^4c_{11}+y_1y_2^2c_8+(k^2c_2-C_0k+\omega)y_1+sy_2=0, \\ y_{2,zzzzzz}d_6+(15k^2d_6+d_4)y_{2,zzzz}+(75k^4d_6+6k^2d_4+d_2)y_{2,zz}+y_2^7d_{15}+ \\ +3y_2^5y_1^2d_{14}+y_2^5d_{11}+3y_2^3y_1^4d_{13}+y_2^3y_1^2d_{10}+y_2^3d_8+61y_2k^6d_6+y_2y_1^6d_{12}+ \\ +5y_2k^4d_4+y_2y_1^4d_9+y_2y_1^2d_7+(k^2d_2-C_0k+\omega)y_1+ry_1=0. \end{gathered} \tag{3}$$

We reduce the system (3) to one equation using the compatibility conditions. Then we look for the solution of the first equation of this system (3) in the form $R(z)=\gamma_0+\gamma_1F(z)$, supposing, without loss of generality, that $\gamma_0=0$. R(z) is a new function, which equals: $R(z)=\frac{y_1(z)}{A_1}=\frac{y_2(z)}{B_1}$. $F(z)$ is a solution of the equation $F_z^2=F^2(1-\chi F^2)$ and has the form:

$$F(z)=\frac{4ae^{\alpha(z-z_0)}}{4a^2e^{2\alpha(z-z_0)}+\chi} \tag{4}$$

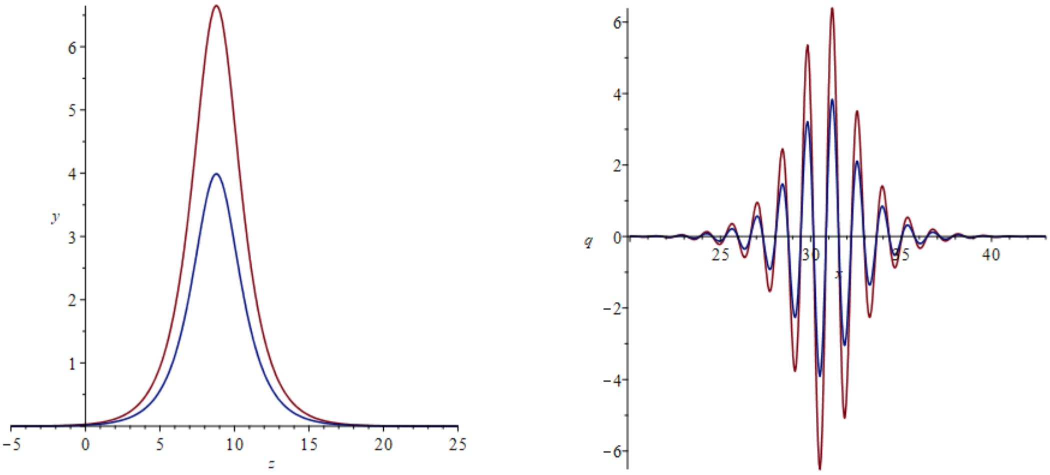
Substituting $R(z)=\gamma_0+\gamma_1F(z)$ into the first equation of the system (3) we obtain the solitary wave solutions:

$$y_1(z)=\frac{4aA_1}{4a^2e^{\alpha z}+\chi e^{-\alpha z}},\quad y_2(z)=\frac{4aB_1}{4a^2e^{\alpha z}+\chi e^{-\alpha z}}. \tag{5}$$

Then, the solution of the initial system (1) can be written in the form:

$$\begin{gathered} u(x,t)=\frac{4a\gamma_1A_1e^{i(-kx+\omega t+\theta_0)}}{4a^2e^{\alpha(x-C_0t-z_0)}+\chi e^{-\alpha(x-C_0t-z_0))}}, \\ v(x,t)=\frac{4a\gamma_1B_1e^{i(-kx+\omega t+\theta_0)}}{4a^2e^{\alpha(x-C_0t-z_0)}+\chi e^{-\alpha(x-C_0t-z_0))}}. \end{gathered} \tag{6}$$

This research was supported by Russian Science Foundation (RSF) no. 18-11-00209.



(a) (b)

Fig. 2: Solitary waves $y_1(z)$ and $y_2(z)$ (b), real parts of u(x,t) and v(x,t) (b) with $a = 1.5, A_1 = 1.0, B_1 = 0.6, k = 4.5, C_0 = 1.0, \alpha = 0.7, \theta_0 = 1.0, z_0 = 12.0, \gamma_1 = 1.45, \chi = 0.1, \omega = 2.28$

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SOLITARY WAVE SOLUTIONS OF THE COUPLED NONLINEAR SCHRÖDINGER EQUATION WITH CUBIC-QUINTIC-SEPTIC FORM OF NONLINEARITY

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The system of equations for describing the propagation of optical pulses in Bragg gratings with cubic nonlinearity is considered. The solution for traveling wave reduction of considered equations with some constraints on parameters is found. Bragg solitons or gap solitons arise in nonlinear optical media with a weakly varying periodic refractive index [1]. Optical fiber with Bragg gratings is widely used in telecommunication systems [2].

Let us consider the coupled nonlinear Schrödinger equation in fiber Bragg gratings with polynomial nonlinearity [3, 4, 5]

$$\begin{aligned} &iq_t + a_1 r_{xx} + \left(b_1|q|^2 + c_1|r|^2\right) q + \left(\xi_1|q|^4 + \eta_1|q|^2|r|^2 + \zeta_1|r|^4\right) q \\ &+ \left(l_1|q|^6 + m_1|q|^4|r|^2 + n_1|q|^2|r|^4 + p_1|r|^6\right) q + i\alpha_1 q_x + \beta_1 r = 0, \end{aligned} \tag{1}$$

$$\begin{aligned} &ir_t + a_2 q_{xx} + \left(b_2|r|^2 + c_2|q|^2\right) r + \left(\xi_2|r|^4 + \eta_2|r|^2|q|^2 + \zeta_2|q|^4\right) r \\ &+ \left(l_2|r|^6 + m_2|r|^4|q|^2 + n_2|r|^2|q|^4 + p_2|q|^6\right) r + i\alpha_2 r_x + \beta_2 q = 0, \end{aligned} \tag{2}$$

where $q(x,t)$ and $r(x,t)$ are the amplitudes of the forward- and backward-propagating waves, $a_j, b_j, c_j, \xi_j, \eta_j, \zeta_j, l_j, m_j, n_j, p_j, \alpha_j, \beta_j (j = 1, 2)$ are parameters of the optical system. We look for exact solution of the system (1), (2) in the form

$$q(x,t) = y_1(z)\, \mathrm{e}^{i(\kappa x - \omega t + \theta)},\ r(x,t) = y_2(z)\, \mathrm{e}^{i(\kappa x - \omega t + \theta)}, \quad z = x - C_0 t, \tag{3}$$

where κ, ω, θ and C_0 are arbitrary constants. After substitution (3) into (1) and (2) we have the system of fourth equations for real and imaginary parts of the system (1), (2) in the form

$$\begin{aligned} &a_1 y_2'' + l_1 y_1^7 + m_1 y_1^5 y_2^2 + n_1 y_1^3 y_2^4 + p_1 y_1 y_2^6 + \eta_1 y_1^3 y_2^2 + \xi_1 y_1^5 \\ &+ \zeta_1 y_1 y_2^4 - \kappa^2 a_1 y_2 + b_1 y_1^3 + c_1 y_1 y_2^2 - \kappa\alpha_1 y_1 + \omega y_1 + \beta_1 y_2 = 0, \end{aligned} \tag{4}$$

$$\begin{aligned} &a_2 y_1'' + l_2 y_2^7 + m_2 y_1^2 y_2^5 + n_2 y_1^4 y_2^3 + p_2 y_1^6 y_2 + \eta_2 y_1^2 y_2^3 + \xi_2 y_2^5 \\ &+\zeta_2 y_1^4 y_2 - \kappa^2 a_2 y_1 + b_2 y_2^3 + c_2 y_1^2 y_2 - \kappa\alpha_2 y_2 + \omega y_2 + \beta_2 y_1 = 0, \end{aligned} \tag{5}$$

$$2\kappa a_1 y_2' - (C_0 - \alpha_1)\, y_1' = 0, \tag{6}$$

$$2\kappa a_2 y_1' - (C_0 - \alpha_2)\, y_2' = 0. \tag{7}$$

If $4\, a_1\, a_2 \kappa^2 \neq (\alpha_2 - C_0)\,(\alpha_1 - C_0)$ then the system (6), (7) has the solution $y_1 = const, y_2 = const$ and this case is not interesting. If

$$\frac{2\kappa a_1}{C_0 - \alpha_2} = \frac{C_0 - \alpha_1}{2\kappa a_2}, \tag{8}$$

then

$$y_2 = \sigma y_1 + C, \tag{9}$$

where C is the constant of integration and $\sigma = \frac{\alpha_1 - v}{2\, a_1\, \kappa}$. Let the integration constant $C = 0$. After substituting the expression (9) into the system (4), (5), multiplying this equations by y_1' and integrating over z we have the system (4), (5) in the form

$$\begin{aligned} 4a_1\sigma y_1'^2 &+ \left(\sigma^6 p_1 + \sigma^4 n_1 + \sigma^2 m_1 + l_1\right) y_1^8 + \left(\frac{4}{3}\sigma^4\zeta_1 + \frac{4}{3}\sigma^2\eta_1 + \frac{4}{3}\xi_1\right) y_1^6 \\ &+ \left(2\sigma^2 c_1 + 2b_1\right) y_1^4 + \left(4\sigma\beta_1 + 4\omega - 4\kappa^2\sigma a_1 - 4\kappa\alpha_1\right) y_1^2 + C_1 = 0, \end{aligned} \tag{10}$$

$$\begin{aligned} 4a_2 y_1'^2 &+ \left(\sigma^7 l_2 + \sigma^5 m_2 + \sigma_3^2 n_2 + \sigma p_2\right) y_1^8 + \left(\frac{4}{3}\sigma^5\xi_2 + \frac{4}{3}\sigma^3\eta_2 + \frac{4}{3}\sigma\zeta_2\right) y_1^6 \\ &+ \left(2\sigma^3 b_2 + 2\sigma c_2\right) y_1^4 + \left(4\omega\sigma + 4\beta_2 - 4\kappa^2 a_2 - 4\kappa\sigma\alpha_2\right) y_1^2 + C_2 = 0. \end{aligned} \tag{11}$$

The balancing number of the system (10), (11) is $p = \frac{1}{3}$. Assuming $y_1(z) = y(z)^{\frac{1}{3}}$ in equations (10), (11) we obtain

$$A_1 y^{10/3} + B_1 y^{8/3} + C_1 y^{4/3} + E_1 y^2 + F_1 y^4 + G_1 y^2 = 0, \tag{12}$$

$$A_2 y^{10/3} + B_2 y^{8/3} + C_2 y^{4/3} + E_2 y^2 + F_2 y^4 + G_2 y^2 = 0, \tag{13}$$

where

$$A_1 = \frac{4}{3}\sigma^4\zeta_1 + \frac{4}{3}\sigma^2\eta_1 + \frac{4}{3}\xi_1,\ A_2 = \frac{4}{3}\sigma^5\xi_2 + \frac{4}{3}\sigma^3\eta_2 + \frac{4}{3}\sigma\zeta_2,$$
$$B_1 = 2\sigma^2 c_1 + 2b_1,\ B_2 = 2\sigma\left(\sigma^2 b_2 + c_2\right),\ E_1 = \frac{4a_1\sigma}{9},\ E_2 = \frac{4a_2}{9},$$
$$F_1 = \sigma^6 p_1 + \sigma^4 n_1 + \sigma^2 m_1 + l_1,\ F_2 = \sigma\left(\sigma^6 l_2 + \sigma^4 m_2 + \sigma^2 n_2 + p_2\right),$$
$$G_1 = \left(-4\kappa^2 a_1 + 4\beta_1\right)\sigma - 4\kappa\alpha_1 + 4\omega,\ G_2 = \left(-4\kappa\alpha_2 + 4\omega\right)\sigma - -4\kappa^2 a_2 + 4\beta_2. \tag{14}$$

We can integrate the system (12), (13) under the constraints on parameters

$$A_1 = 0,\ A_2 = 0,\ B_1 = 0,\ B_2 = 0,\ C_1 = 0,\ C_2 = 0 \tag{15}$$

and the compatibility condition

$$\frac{E_1}{E_2} = \frac{F_1}{F_2} = \frac{G_1}{G_2}. \tag{16}$$

The system (12),(13) under conditions (15),(16) has the general solution in the form

$$y(z) = \frac{-4\,E_1 G_1 \mathrm{e}^{\frac{\sqrt{-E_1 G_1}(z-z_0)}{E_1}}}{-4E_1^2 G_1 F_1 \mathrm{e}^{\frac{-2\sqrt{-E_1 G_1} z_0}{E_1}} + \mathrm{e}^{\frac{2\sqrt{-E_1 G_1} z}{E_1}}}. \tag{17}$$

We found the exact solution of the system of equations (1), (2) in the form

$$q(x,t) = y(z)^{\frac{1}{3}}\,\mathrm{e}^{i(\kappa x - \omega t + \theta)},\ r(x,t) = \sigma y(z)^{\frac{1}{3}}\,\mathrm{e}^{i(\kappa x - \omega t + \theta)}, z = x - C_0 t, \tag{18}$$

with conditions on parameters (8), (15), (16).

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NONLINEAR DYNAMICAL PROCESSES DECRIBED BY THE KURAMOTO-SIVASHINSKY EQUATION

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Generalized Kuramoto-Sivashinsky equation describes a number of nonlinear physical processes. We investigate the change of its dynamics by computing the largest Lyapunov exponent as a function of the dispersive parameter. To transform PDE into the system of ODEs we use a five-point stencil. It is shown how the system's behavior transitions from chaotic to periodic as the bifurcation parameter increases.

Generalized Kuramoto-Sivashinsky equation has the form

$$u_t + u^m u_x + u_{xx} + \beta u_{xxx} + u_{xxxx} = 0, \qquad (1)$$

where β is the constant coefficient and m is the degree of nonlinearity. It describes a number of physical processes, such as waves in chemical reactions [1] and flame front propagation [2].

We choose periodic boundary conditions $u(L,t) = u(0,t),\ \ u_x(L,t) = u_x(0,t)$, and so forth. The initial condition is

$$u(x,0) = sin(\frac{\pi x}{L}), \qquad (2)$$

where L is the size of the domain.

To approximate spatial derivatives of the PDE (1) we use the five-point stencil. Benettin algorithm [4] is applied to compute largest Lyapunov exponents of the obtained ODE system, consisting of N equations, where N is the number of points within the uniform grid.

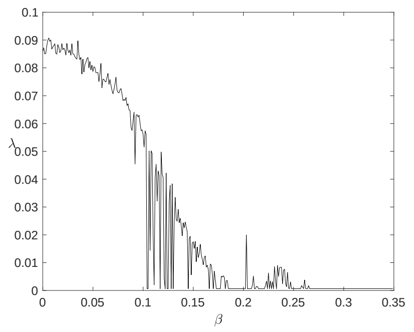
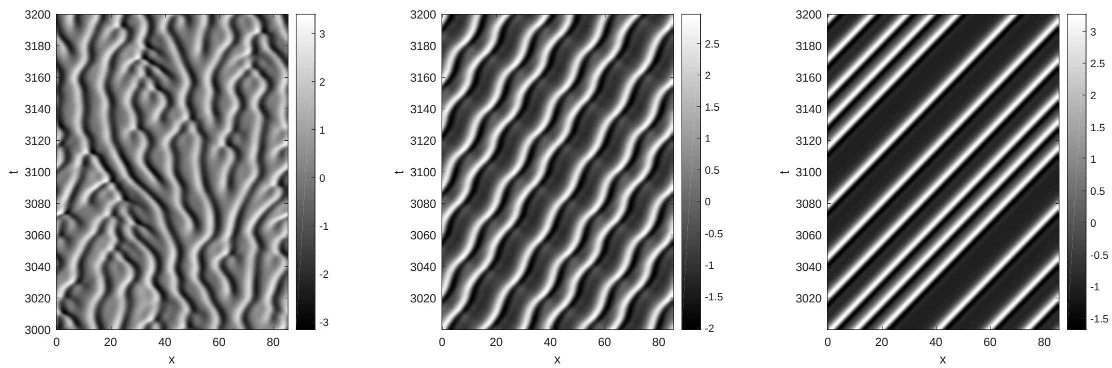
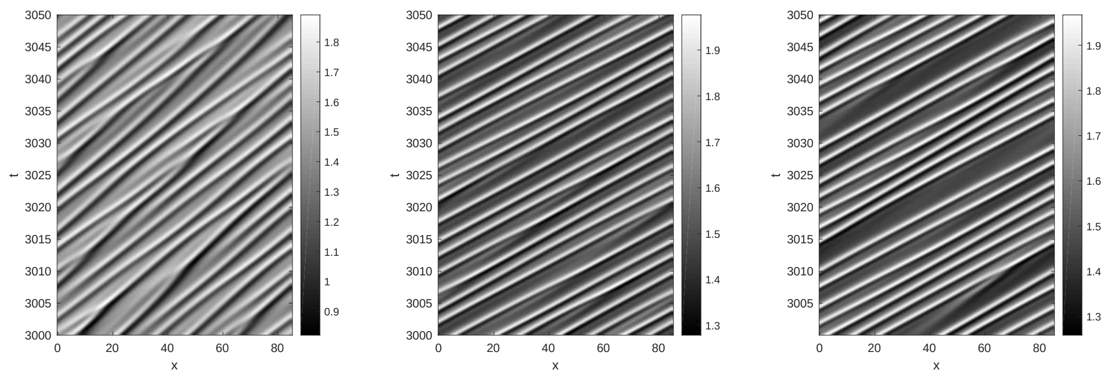


Fig. 1: Largest Lyapunov exponent of the equation (1) as a function of β for $L = 85$ for m=1(left) and for m=3(right).



$a)\ \beta = 0$ $b)\ \beta = 0.2$ $c)\ \beta = 0.4$



$a)\ \beta = 0$ $b)\ \beta = 0.15$ $c)\ \beta = 0.3$

Fig. 2: Solution of the equation (1) as a function of x and t for β and $L = 85$ for m=1(top) and m=3(bottom).

Lyapunov exponent as a function of β for $m = 1$ and $m = 3$ is presented on the fig. 1 and fig. 2 show how the solution of (1) gradually changes its behavior from chaotic to periodic as the parameter β increases.

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PAINLEVÉ ANALYSIS AND EXACT SOLUTION TO THE TRAVELING WAVE REDUCTION OF NONLINEAR DIFFERENTIAL EQUATIONS FOR DESCRIBING PULSE IN OPTICAL FIBER

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Two nonlinear partial differential high-order equations are considered. They are used for describing propagation pulses in optical fibers. The Painlevé analysis for traveling wave reduction of the equations is completed. As a result of Painlevé test the condition on some parameters of the models are obtained. For constructing the exact solution, simplest equations method is used. Periodic and solitary wave solutions are found.

In addition to the popular model of propagation optical pulses [1-3], there are some new models. In particular the method of constructing the high-order partial differential equations for description optical pulses are proposed in paper [4]. We study two equations that are constructed in article [4].

Fourth-order equations with third- , fifth-degree nonlinearity and non-local nonlinearity:

$$\begin{gathered} iu_t + \alpha u_{xx} + i\beta u_{xxx} + u_{xxxx} \\ +\mu|u|^2u + \nu|u|^4u + \varkappa|u|^2u_{xx} + i\frac{\beta\varkappa}{2}|u|^2u_x = 0, \end{gathered} \tag{1}$$

$\alpha,\ \beta,\ \mu,\ \nu,\ \varkappa$ are the parameters of equation (1).

Sixth-order equations with third- , fifth- , seventh-degree nonlinearity and non-local nonlinearity:

$$\begin{gathered} iu_t + \alpha u_{xx} + i\beta u_{xxx} + \chi u_{xxxx} + i\delta u_{xxxxx} + u_{xxxxxx} \\ +\mu|u|^2u + \nu|u|^4u + \varkappa|u|^6u + \rho|u|^2u_{xx} + i\frac{\delta\rho}{3}|u|^2u_x = 0, \end{gathered} \tag{2}$$

$\alpha,\ \beta,\ \delta,\ \mu,\ \nu,\ \rho,\ \varkappa,\ \chi$ are the parameters of equation (2).

We consider equations using traveling wave reduction

$$\mathrm{u(x,t) = y(z)\exp\left(i(\psi(z) - \omega\, t)\right), \qquad z = x - C_0\, t.} \tag{3}$$

Using traveling wave variables (3), we obtain a systems of ordinary differential equations for the imaginary and real parts. These systems do not pass the Painlevé test, but we get some conditions under which the models is simplified. For the traveling wave reduction of equation (1) we obtain

$$C_0 = -\frac{1}{2}\alpha\beta - \frac{1}{8}\beta^3, \quad \psi_z = -\frac{\beta}{4}. \tag{4}$$

For the reduction of equation (2) we get the following conditions

$$C_0 = -\frac{1}{81}\delta^5 - \frac{1}{27}\chi\delta^3 - \frac{1}{3}\alpha\delta, \quad \beta = \frac{5}{27}\delta^3 + \frac{2}{3}\chi\delta, \quad \psi_z = -\frac{\delta}{6}. \tag{5}$$

Using condition (4) we get the traveling wave reduction of equation (1) in the form

$$\begin{gathered} y_{zzzz} + \left(\alpha + \frac{3}{8}\beta^2\right)y_{zz} + \left(\omega + \frac{\alpha\,\beta^2}{16} + \frac{5}{256}\,\beta^4\right)y + \left(\mu + \frac{1}{16}\varkappa\beta^2\right)y^3 \\ +\nu y^5 + \varkappa y^2 y_{zz} = 0. \end{gathered} \tag{6}$$

Under condition (5) the traveling wave reduction of equation (2) takes the form

$$\begin{gathered} y_{zzzzzz} + \left(\frac{5}{12}\delta^2 + \chi\right)y_{zzzz} + \left(\frac{25}{432}\delta^4 + \frac{1}{6}\chi\delta^2 + \alpha\right)y_{zz} + \rho y^2 y_{zz} \\ +\left(\frac{1}{36}\alpha\delta^2 + \frac{5}{1296}\chi\delta^4 + \omega + \frac{61}{46656}\delta^6\right)y + \left(\mu + \frac{1}{36}\delta^2\rho\right)y^3 \\ +\nu y^5 + \varkappa y^7 = 0. \end{gathered} \tag{7}$$

Using the method of simplest equations [5] for equations (6) and (7), we construct the exact solutions of (6), (7). We get the periodic wave solution in the form

$$y(z) = A\sqrt{\frac{3d}{3\wp\left(z - z_0, \frac{4}{3}c^2 - 4ad, \frac{4}{3}acd - \frac{8}{27}c^3\right) - c}}, \tag{8}$$

where z_0, a are arbitrary constants, A, c, d - depend on the parameters of the equations. And we obtain a solitary wave solution in the form

$$y(z) = A\frac{4ce^{\sqrt{c}(z-z_0)}}{e^{2\sqrt{c}(z-z_0)} - 4ac}, \tag{9}$$

where z_0, a are arbitrary constants, A, c - depend on the parameters of the equations.

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SPATIALLY INHOMOGENEOUS EQUILIBRIUM STATES OF THE CAHN-HILLIARD EQUATION

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The Cahn–Hilliard equation was obtained in [1] as a model describing the boundary interaction in chemical kinetics with allowance for spatial effects. Subsequently, it was used in a number of branches of physics, for example, hydrodynamics, [2]. Usually, in the literature devoted to its analysis, one considers its “one-dimensional” version, when the unknown function depends only on the variable t and one spatial variable x (see, [3-4] and the list of references therein).

A more natural version of the Cahn–Hilliard equation is the one where the unknown function depends on two spatial variables. In this case, after normalizing t, x, y and the function $u = u(t, x, y)$, the Cahn–Hilliard equation can be written as

$$u_t = -\Delta_\mu^2 u - b\Delta_\mu u - b_2\Delta_\mu(u^2) + \Delta_\mu(u^3), \tag{1}$$

where $x, y \in [0, \pi], 0 < \mu \leq 1, \Delta_\mu = u_{xx} + \mu u_{yy}$, and $\Delta_1 = \Delta$ is the Laplace operator. For this version of the Cahn–Hilliard equation, two boundary value problems are studied, namely

$$u|_{x=0,x=\pi} = \Delta_\mu u|_{x=0,x=\pi} = u|_{y=0,y=\pi} = \Delta_\mu u|_{y=0,y=\pi} = 0 \tag{2}$$

$$u_x|_{x=0,x=\pi} = u_{xxx}|_{x=0,x=\pi} = u_y|_{y=0,y=\pi} = u_{yyy}|_{y=0,y=\pi} = 0. \tag{3}$$

An analysis of the boundary value problem (1), (2) showed that for $b < 1 + \mu$ it has a homogeneous equilibrium state $u = 0$ is asymptotically stable and unstable if $b > 1 + \mu$. For $b = 1 + \mu + \gamma\varepsilon, \varepsilon\ in(0, \varepsilon_0)$ for the boundary value problem (1), (2), we have a case close to the critical simple zero value of the stability spectrum. In this case, for all sufficiently small $\varepsilon\ in(0, \varepsilon_0)$, the boundary-value problem realizes a variant of transcritical bifurcation if $b_2 \neq 0$, or bifurcations of type “plug” if $b_2 = 0$.

A more complex bifurcation problem arises in the analysis of the boundary value problem (1), (3): it has a family of equilibrium states $S_\alpha : u(t, x) = \alpha, \alpha \in \mathbb{R}$.

Further, the main results of the analysis of the boundary value problem (1), (3) are formulated for the case $b_2 = 0$. For $b_2 \neq 0$ these statements take on a more cumbersome form.

Let $\mu \leq 1, b_2 = 0$ and S_α be such that $\alpha^2 > (b-\mu)/3$. Then, S_α is stable and unstable if $\alpha^2 < (b-\mu)/3$. When $\alpha^2(\varepsilon) = (b-\mu-\gamma\varepsilon)/3$, where $\gamma = \pm 1, \varepsilon \in (0, \varepsilon_0), 0 < \varepsilon_0 << 1$, a critical case is realized in the problem of stability of the equilibrium state: for $\mu < 1$ twice the eigenvalue of the stability spectrum, and for $\mu = 1(\mu = 1 - \gamma_2\varepsilon)$ three times zero eigenvalue.

Let $\mu < 1, 0 < \varepsilon_0 << 1$. In this case, the boundary value problem (1), (2) for each $\varepsilon \in (0, \varepsilon_0)$ has spatially heterogeneous solutions

$$u(y,\varepsilon) = \alpha(\varepsilon) \pm 2\varepsilon^{1/2}\Big|2b - 5\mu\Big|^{-1/2}\cos y + o(\varepsilon^{1/2}).$$

Such solutions are stable if $b \in (\mu, 5\mu/2)$ and unstable if $b > 5\mu/2$.

For $\mu = 1$ $(\mu = 1 - \gamma_2\varepsilon, \gamma_2 \in \mathbb{R})$ a bifurcation problem of codimension 2 arises and spatially inhomogeneous equilibrium states appear that depend only on x, only on y, and also from x, y at the same time. The question of their stability is investigated and asymptotic formulas are obtained for them. To study bifurcation problems, the method of integral manifolds and normal forms was used.

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ANALYSIS OF BIFURCATIONS OF SPATIALLY INHOMOGENEOUS SOLUTIONS OF A NONLINEAR PARABOLIC EQUATION WITH THE OPERATOR OF ROTATION OF THE SPATIAL ARGUMENT AND DELAY

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For a differential equation with a lagging argument

$$u_t(\rho,\phi,t)+u(\rho,\phi,t)=D\Delta_{\rho\phi}u(\rho,\phi,t)+K(1+\gamma cos(u_\theta(\rho,\phi,t-T))) \quad (1)$$

relatively to the function $u(\rho,\phi,t+s)$, specified in polar coordinates $0\le\rho\le R, 0\le\phi\le 2\pi$ $(R>0)$ and $t\ge 0, -T\le s\le 0$ $(T>0)$, in which $\Delta_{\rho\phi}$ is the Laplace operator inpolar coordinates, $u_\theta(\rho,\phi,t)\equiv u(\rho,(\phi+\theta)mod(2\pi),t)$ $(0\le\theta<2\pi)$ is the rotation operator of the spatial argument, D, K are positive constants, $0<\gamma<1$, in the area $\bar{K}_R\times\mathbb{R}^+$, where circle $\bar{K}_R=\{(\rho,\phi): 0\le\rho\le R, 0\le\phi\le 2\pi\}$, $\mathbb{R}^+=\{t: 0\le t<\infty\}$, an initial-boundary value problem of the form is considered

$$u_\rho(R,\phi,t)=0,\quad u(\rho,0,t)=u(\rho,2\pi,t),\quad u_\phi(\rho,0,t)=u_\phi(\rho,2\pi,t),$$

$$u(\rho,\phi,t+s)|_{t=0}=u_0(\rho,\phi,s)\in H_0(K_R;-T,0). \quad (2)$$

In (2), the space of initial conditions $H_0(K_R;-T,0)=\{u(\rho,\phi,s): u(\rho,\phi,s)\in C(\bar{K}_R\times[-T,0]), u(\rho,0,s)=u(\rho,2\pi,s)$, for each s $u(\rho,\phi,s)\in \overset{\circ}{W}{}_2^2(K_R)\}\}$, where the space of the functions $\in\overset{\circ}{W}{}_2^2(K_R)\}$ is obtained by closing the set of functions $\{u(\rho,\phi): u(\rho,\phi)\in C^2(\bar{K}_R), u_\rho(R,\phi)=0, u(\rho,0)=u(\rho,2\pi), u_\phi(\rho,0)=u_\phi(\rho,2\pi)\}$ in the metric of the space of functions $W_2^2(K_R)$.

The phase space of the initial-boundary value problem (1)-(2) is the space
$H(K_R;-T,0)=\{u(\rho,\phi,s): u(\rho,\phi,s)\in L_2(K_R)$ for each $-T\le s\le 0, ||u(\rho,\phi,s)||_{L_2}\in C([-T,0])\}$, the norm of which is defined as $||u(\rho,\phi,s)||_H=\max_s||u(\rho,\phi,s)||_{L_2}$. By the area of the definition of the right-hand side of equation $H_0(K_R;-T,0)$. The norm in $H_0(K_R;-T,0)$ is defined as $||u(\rho,\phi,s)||_{H_0}=\max_s||u(\rho,\phi,s)||_{W_2^2}$.

By the solution of the initial-boundary value problem (1)-(2), defined for $t > 0$, we mean the function $u(\rho, \phi, t + s) \in H_0(K_R; -T, 0)$ (for each $t > 0$), continuously differentiable by t when $t > 0$, turning equation (1) into an identity and satisfying the initial conditions (2).

The work investigates the dynamics of homogeneous equilibrium states and their stability independing on the parameters of equation (1). In the plane of the main control parameters(amplification coefficient K and rotation angle θ) using the D-partition method and its special parametrization stability (instability) areas of heterogeneous equilibrium states are constructed. The dynamics of stability areas is investigated depending on the value of the delay and other parameters of the initial-boundary value problem. Possible mechanisms of loss of stability by homogeneous equilibrium states are revealed. Using the center manifold method and bifurcation theory possible bifurcations of spatially inhomogeneous self-oscillatory solutions and also their stability are explored. The dynamics of such solutions in the vicinity of the boundary of the stability area in the plane of control parameters is studied.

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TOPOLOGICAL INVARIANTS OF MONGEAMPRE GRASSMANNIANS

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We will give a description of the integral MongeAmpère Grassmannians $IE_\omega(x)$ for MongeAmpère operators with 2 and 3 independent variables for a non-degenerate effective MongeAmpère form ω. We shall give an account to the theory of the characteristic classes based on the integral Grassmannians $IE_\omega(x)$ and show that even for $n = 3$ the topological structure of such Grassmannians is interesting and complicated. Their topological structure (decomposition to various regularity strata) is different for even (easy case) and for odd (highly non-trivial) values of n. We find a relation with some exciting object (Cayley affine cubic surface) which deserves future explorations.

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BIFURCATION SCENARIO IN THE AMPLITUDE SYSTEM OF TWO COUPLED OSCILLATORS

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In this work, Bazykin population model with a delay [1], [2], represented by a system of diffusely coupled oscillators is considered:

$$\begin{aligned} \dot{v}_1 &= -\Big(\tfrac{\pi}{2}+\varepsilon\Big)v_1(t-1)(1+v_1)^2+\varepsilon d(v_2-v_1),\\ \dot{v}_2 &= -\Big(\tfrac{\pi}{2}+\varepsilon\Big)v_2(t-1)(1+v_2)^2+\varepsilon d(v_1-v_2). \end{aligned} \tag{1}$$

It is assumed that the coupling between oscillators in system (1) is weak, that is, coefficient $d>0$ is proportional to a small parameter $0<\varepsilon\ll 1$. Note that if $\varepsilon=0$, in the stability spectrum of the equilibrium state $(0,0)^T$ there is a pair of purely imaginary eigenvalues $\lambda=\pm i\frac{\pi}{2}$ of multiplicity 2, which two linearly independent eigenfunctions are corresponding.

To study the dynamics of system (1) we apply the standard method of normal forms [3], using the following replacement: $v(t,\varepsilon)=\sqrt{\varepsilon}v_{0j}(t,\tau)+\varepsilon v_{1j}(t,\tau)+\varepsilon^{3/2}v_{2j}(t,\tau)+\ldots$, where $v_{0j}(t,\tau)=z_j(\tau)e^{i\frac{\pi}{2}t}+\overline{z}_j(\tau)e^{-i\frac{\pi}{2}t}$, $z_j(\tau)$ — complex–valued functions of a slow–time variable $\tau=\varepsilon t$, $j=1,2$. At the third step of the algorithm we obtain the following normal form written in amplitude and phase variables from the solvability conditions for $v_{2j}(t,\tau)$ in the class of 4–periodic by t functions:

$$\begin{aligned} \xi_1' &= (1-D\cos\delta-\xi_1^2)\xi_1+D\xi_2\cos(\varphi+\delta),\\ \xi_2' &= (1-D\cos\delta-\xi_2^2)\xi_2+D\xi_1\cos(\varphi-\delta),\\ \varphi' &= -b(\xi_2^2-\xi_1^2)-D\Big(\tfrac{\xi_1}{\xi_2}\sin(\varphi-\delta)+\tfrac{\xi_2}{\xi_1}\sin(\varphi+\delta)\Big). \end{aligned} \tag{2}$$

In system (2) functions $\xi_1(s)$ and $\xi_2(s)$ represent slowly changing amplitudes of cycles, and $\varphi(s)$ — is the difference of their phases. Parameters are selected as follows: $b=2(7+2\pi)/(7\pi-8)$, $D=d\sqrt{\pi^2+4}/\pi$, $\delta=-\arctan(\pi/2)$.

If the parameter $b=b_H=(\pi+6)/(3\pi-2)$, the system (2) fully corresponds to the system obtained in [4] as the normal form of the system of diffuse weakly coupled Hutchinson equations. In works [4]–[6] an analysis of its qualitative behavior is presented.

In this article, we investigate the dynamics of the normal form (2) for the parameter $b = 2(7+2\pi)/(7\pi-8)$. Consider the script of phase transformations in the amplitude system (2), obtained with this value of b when the parameter D is changed:

1. System (2), in contrast to the situation $b = b_H$, does not have subcritical stable regimes. For $D > D_{\text{cr.}} = -\cos\delta + b\sqrt{1-\cos\delta^2} \approx 1.06472$ the homogeneous equilibrium state $(1,1,0)^T$ is globally asymptotically stable.

2. At $D = D_{\text{cr.}}$ a pair of symmetric stable equilibrium states $A = (\xi_1^*(D), \xi_2^*(D),$
$\alpha^*(D))^T$ and $B = (\xi_2^*(D), \xi_1^*(D), -\alpha^*(D))^T$ branches off from a homogeneous one, inheriting its stability. Note that $\xi_1^*(D), \xi_2^*(D) \to 1$, $\alpha^*(D) \to 0$ as $D \to D_{\text{cr.}}$.

3. When $D < D_{\pi 1} = 1/(2\cos\delta) \approx 0.931$ an unstable equilibrium state $(\xi^*, \xi^*, \pi)^T$, where $\xi^* = \sqrt{1-2D\cos\delta}$, appears (corresponds to oscillations in the antiphase of the initial system). Note that in the case $b = b_H$ antiphase oscillations occur at $D_{\pi 1} > D_{\text{cr.}}$.

4. At $D = D_C \approx 0.7547$ symmetric stable equilibrium states A and B gently lose stability with the birth of stable cycles C_A and C_B (Andronov–Hopf bifurcation).

5. Further reduction of the parameter D leads to the fact that the stable cycles C_A and C_B increase in size until, at $D = D_S \approx 0.6391$, they close at the point $\xi_1 = \xi_2 = 1$, $\alpha = 0$ (reverse separatrix splitting bifurcation). As a result, two cycles C_A and C_B are combined into one C_U.

6. At $D = D_{\pi 2} = 1/(4\cos\delta) \approx 0.4655$ the unstable cycle C_Π branches off from the unstable state of equilibrium $(\xi^*, \xi^*, \pi)^T$.

7. At $D = D_{\pi 3} \approx 0.4647$ the unstable cycle C_Π merges with the stable cycle C_U and disappears.

8. For $0 < D < D_{\pi 3}$ the system has a unique, globally stable equilibrium state $(\xi^*, \xi^*, \pi)^T$, corresponding to antiphase oscillations of the initial system.

The last four phase transformations exactly repeat the bifurcations received in the normal form of the system of diffusely weakly coupled Hutchinson equations (see, for example, [4]–[6]).

Dynamic properties of the normal form (2) are investigated. A complete script of phase rearrangements occurring in the system when the diffusion parameter changes is given. It should be particularly noted that in this case, other stable regimes of the normal form cannot coexist with a homogeneous equilibrium state and for $D > D_{\text{cr.}}$ a homogeneous state of equilibrium is globally stable.

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QUANTISATION IDEALS OF NONABELIAN INTEGRABLE SYSTEMS

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In my talk I'll show a new approach to the problem of quantisation of dynamical systems, introduce the concept of quantisation ideals and provide meaningful examples. Traditional quantisation theories start with classical Hamiltonian systems with functions taking values in commutative algebras and then study there non-commutative deformations such that the commutators of observables tend to the corresponding Poisson brackets as the (Planck) constant of deformation goes to zero. I am proposing to depart from systems defined on a free associative algebra $\mathfrak{A}$. In this approach the quantisation problem is reduced to a description of two-sided ideals $\mathfrak{I} \subset \mathfrak{A}$ which define the commutation relations (the *quantisation ideals*) in the quotient algebras $\mathfrak{A}_{\mathfrak{I}} = \mathfrak{A} \diagup \mathfrak{I}$ and which are invariant with respect to the dynamics of the system. Surprisingly this idea works rather efficiently and in a number of cases I have been able to quantise the system, i.e to find commutation relations for the system.

To illustrate this approach I'll consider the quantisation problem for the non-abelian Bogoyavlensky N–chains

$$\frac{du_n}{dt} = \sum_{k=1}^{N} (u_{n+k}u_n - u_n u_{n-k}), \qquad n \in \mathbb{Z} \tag{1}$$

where functions u_k are elements of a free associative algebra $\mathfrak{A} = \mathbb{K}\langle \dots u_{-1}, u, u_1, \dots \rangle$ over a zero characteristic field of constants $\mathbb{K}$ (the case $N = 1$ corresponds to the well known Volterra chain). I will show that system (1) admits a quantisation with commutation relations

$u_n u_{n+k} = \alpha u_{n+k} u_n$, for $1 \le k \le N$, and $u_n u_{n+k} = u_{n+k} u_n$, for $k > N$, $\alpha \in \mathbb{K}^{\times}$.

Other examples will also be presented.

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THE GROWTH OF POLYNOMIAL LIE-REINHART ALGEBRAS

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Consider a commutative algebra A over a commutative unital ring R. The pair $(A, \mathcal{L})$ is called a Lie-Rinehart algebra [1] if

1) $\mathcal{L}$ is a Lie algebra over the ring R which acts on A by left derivations

$$X(ab) = X(a)b + aX(b), \forall a, b \in A, \forall X \in \mathcal{L};$$

2) the Lie algebra $\mathfrak{g}$ is a A-module.

The pair $(A, \mathcal{L})$ must satisfy the compatibility conditions

$$\begin{aligned} [X, aY] = X(a)Y + a[X, Y], \forall X, Y \in \mathcal{L}, \forall a \in A; \\ (aX)(b) = a(X(b)),\ \forall a, b \in A, \forall X \in \mathcal{L}. \end{aligned} \tag{1}$$

Consider an important subclass [2] of graded Lie-Rinehart algebras $(A, \mathcal{L})$, where $A = R[t_1, t_2, \ldots, t_p]$ is a graded polynomial algebra over R such that

1) $\mathcal{L}$ is a free left module over $R[t_1, t_2, \ldots, t_p]$ of rang N.

2) $\mathcal{L} = \oplus_{i \in \mathbb{Z}} \mathcal{L}_i$ is a $\mathbb{Z}$-graded Lie algebra $[\mathcal{L}_i, \mathcal{L}_j] \subset \mathcal{L}_{i+j},\ i, j \in \mathbb{Z}$, and its grading is compatible with the grading $R[t_1, t_2, \ldots, t_p]$.

$$p(t)L \in \mathcal{L}_{i+deg(p(t))},\ deg(L(q(t)) = deg(q(t)) + i, L \in \mathcal{L}_i.$$

where $p(t), q(t)$ are homogeneous polynomials $R[t_1, t_2, \ldots, t_p]$ of degree $deg(p(t))$ and $deg(q(t))$ respectively. The algebra grading $R[t_1, t_2, \ldots, t_p]$ is defined on generators by the formulas

$$deg(t_1) = m_1, \ldots, deg(t_p) = m_p, m_i \in \mathbb{Z}.$$

We will discuss the growth of a Lie algebra (over R), generated by the left free module $\mathcal{L}$ over $R[t_1, t_2, \ldots, t_p]$. Its growth rate is related to the integrability of some systems of hyperbolic PDE (Klein-Gordon equation) [3,4].

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DEGENERATE RESONANCES IN THE QUASI-PERIODICALLY FORCED DUFFING EQUATION WITH THE ASYMMETRY OF THE POTENTIAL WELL

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Non–conservative quasi-periodic perturbations of two–dimensional nonlinear Hamiltonian systems are studied. In contrast to the previously considered case of Hamiltonian systems with monotonic rotation, we suppose that the eigenfrequency of the unperturbed system has stationary points. If the corresponding phase curve is resonant, then the resonance is called degenerate. Systems where degenerate resonances may occur are relevant to applications. An example is a widespread Duffing type equation with the asymmetry of the potential well. We study the structure of the degenerate resonance zone of the equation and establish the conditions for the existence of resonant quasi–periodic solutions. The problem of oscillations synchronization is considered as well. Numerical results complement the study.

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PERIODIC AND STATIONARY SOLUTIONS OF NONLINEAR REACTION-DIFFUSION PROBLEMS WITH SINGULARLY PERTURBED BOUNDARY CONDITIONS

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We present an extension of asymptotic method of differential inequalities (see [1] and references there in) to new classes of problems with singularly perturbed boundary conditions. We illustrate our results by consideration of boundary and interior layer type stationary solutions of the reaction-diffusion problem

$$\begin{cases} \varepsilon^2 \dfrac{\partial^2 v}{\partial x^2} - \dfrac{\partial v}{\partial t} = f(v, x, \varepsilon), \quad x \in (-1;1), \quad t > 0; \\ \dfrac{dv}{dx}(\mp 1, t) = u^{(\mp)}, \; v(x,0) = v_{init}(x, \varepsilon) \end{cases} \tag{1}$$

and its extension to multidimensional in space variable case. Hear the source function f is sufficiently smooth.

We also present the extension of our results for periodic parabolic problem with singularly perturbed boundary conditions

$$\begin{aligned} &\varepsilon^2 \left(\triangle u - \frac{\partial v}{\partial t} \right) - f(u, x, t, \varepsilon) = 0, \\ &(x,t) \in D_t := \{(x,t) \in R^3 : x \in D, t \in R\}, \\ &\varepsilon \frac{\partial u(x,t,\varepsilon)}{\partial n} = u_\Gamma(x,t), \quad x \in \Gamma, t \in R, \\ &u(x,t,\varepsilon) = u(x, t+T, \varepsilon), \quad x \in \bar{D}, t \in R, \end{aligned} \tag{2}$$

Problems of the types (1) and (2) have a lot of applications (see [2], [3] and references therein). The conditions of the stability or instability stationary solutions are presented. In particular, we prove the stability of stationary solutions with of non monotone boundary layer. The study conducted in this work gives an answer about local and nonlocal attraction domain of the stable stationary solutions or stable periodic solutions. The results are used for the stabilization of unstable solutions of the corresponding Dirichlet problems. They sagest an numerical approach of computing unstable solutions of various applied problems by numerical methods of stationing.

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ASYMPTOTIC SOLUTION OF COEFFICIENT INVERSE PROBLEMS FOR INTERIOR LAYER BURGERS TYPE EQUATIONS WITH MODULAR NONLINEARITY

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Recent results of using asymptotic analysis for asymptotic solving of some classes inverse problems for reaction-diffusion-advection equations are presented. This approach is applied to a new class of time-periodic reaction-diffusion-advection problems with internal transition layers. Particularly, for Burgers-type equation, which has a time-periodic solution of moving front type, asymptotic analysis was applied to solve the inverse problem of restoring some parameters of the original model by known information about the observed solution of the direct problem at a given time interval (period).

These results were illustrated by the following problem

$$\begin{aligned}
&\varepsilon\left(\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial t}\right)+A(u,x,t)\cdot\frac{\partial |u|}{\partial x}=B(u,x,t,\varepsilon),\\
&(x,t)\in\mathcal{D}:=\{x\in(-1,1);\quad t\in R\},\\
&u(-1,t)=u_{\text{left}}(t),\quad u(1,t)=u_{\text{right}}(t),\qquad t\in R,\\
&u(x,0)=u(x,t+T),\qquad x\in[-1,1],\qquad t\in R,
\end{aligned}$$

Functions $A(u,x,t)>0$, $B(u,x,t,\varepsilon)$, $u_{\text{left}}(t)$ and $u_{\text{right}}(t)$ are sufficiently smooth and T-periodic in t. This equation contains so-called modular advection or nonlinearity. Problems of this type have a lot of applications.

The inverse problem is to restore coefficients of reaction or advection in the equation, or boundary regimes, using given information about the observed moving front location.

Main idea of our approach is based on the fact that asymptotic analysis allows to reduce the original problem to a much simpler problems which connect with given accuracy some parameters of the original model to be restored with the observed data. The concept of an asymptotic solution of coefficient inverse problems is introduced. The accuracy of the solution is estimated.

The proposed approach can be applied to sufficiently wide class of problems with boundary and internal layers.

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ON THE DYNAMICS OF CERTAIN HIGHER-ORDER SCALAR DIFFERENCE EQUATION

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We construct the asymptotics as $n \to \infty$ for the solutions of the following $(k+1)$-th order scalar linear difference equation:

$$x(n+k+1) - \frac{k+1}{k} x(n+k) + \Big(\frac{1}{k} + q(n)\Big) x(n) = 0, \quad n \in \mathbb{N}. \quad (1)$$

Here the real function $q(n)$ tends to zero as $n \to \infty$ in an oscillatory way. We now outline the problems that lead to Eq. (1).

First, we note that Eq. (1) is equivalent to the linear delay difference equation

$$\Delta y(n) = -p(n) y(n-k), \quad (2)$$

if

$$\lim_{n\to\infty} p(n) = \frac{k^k}{(k+1)^{k+1}}. \quad (3)$$

Here the symbol Δ stands for the forward difference operator. It is known (see, e.g., [1, 2]) that if function $p(n)$ tends to the limit value in (3) in an oscillatory way the critical case in the oscillation problem for solutions of Eq. (2) occurs.

Another problem concerning Eq. (1) comes from the stability theory. The so called critical case of the stability problem occurs in this equation. Namely, the characteristic polynomial of the unperturbed equation ($q(n) = 0$)

$$L(\lambda) = \lambda^{k+1} - \frac{k+1}{k} \lambda^k + \frac{1}{k}$$

has the multiple root $\lambda_{1,2} = 1$ and all the other roots lie inside the unit circle in $\mathbb{C}$.

In the case $k = 1$ equation (1) may be considered as the difference Schrödinger equation with zero energy and the discrete Wigner–von Neumann type potential. The asymptotic formulae in this situation were constructed in [3].

To construct the asymptotics for solutions of Eq. (1) we use the special asymptotic summation method, proposed in [4]. This method uses the fact that there exists for sufficiently large n the attractive invariant manifold (called critical manifold) in the phase space of linear system that is equivalent to Eq. (1). It is possible to construct the asymptotics for solutions of the latter system lying on this manifold. Due to the attractivity property of the manifold, this allows us to get the asymptotics for solutions of the initial Eq. (1). To simplify the process of constructing the asymptotics we also apply the averaging changes of variables, described in [3].

We illustrate the obtained asymptotic formulae for solutions of Eq. (1) by two examples:

$$q(n) = \frac{(-1)^n}{n^\alpha} \quad \text{and} \quad q(n) = \frac{\sin \omega n}{n^\alpha},$$

where $\alpha > 0$ and $0 < \omega < 2\pi$, $\omega \neq \pi$.

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FROM SOLUTIONS OF YANG–BAXTER EQUATIONS TO HIGHER DIMENSIONAL INTEGRABLE SYSTEMS

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We show how solutions of the set theoretical Yang–Baxter equation can be used as building blocks for a hierarchy of integrable maps in high dimensions. We give several examples of such integrable maps in dimension 3 and we explore their integrability and dynamical properties. These maps can also be seen as numerical interpolating algorithms for integrable vector fields. We generalise this construction using solutions of the set theoretic entwining Yang–Baxter equation and we present new integrable maps.

A NEW CLASS OF INTEGRABLE TWO-COMPONENT SYSTEMS OF HYDRODYNAMIC TYPE

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We consider a class of Hamiltonian systems of hydrodynamic type, whose Hamiltonian densities quadratically depend on one of the field variables. It is shown that such Hamiltonian systems have an infinite set of polynomial conservation laws in this field variable.

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LOCAL DYNAMICS OF CAHN–HILLIARD EQUATION

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The study of kinetics of fibering in binary mixtures with fixed concentration of components is one of the up to date objectives of condensed state physics. Cahn-Hilliard equation is one of the models used while studying spontaneous phase separation (binary) of a substance (an alloy), where the unknown function is a relative concentration of a substance component.

We study the generalized Cahn-Hilliard equation

$$\frac{\partial u}{\partial t} = -\frac{\partial^2}{\partial x^2}\left[\alpha\frac{\partial^2 u}{\partial x^2} + \lambda\frac{\partial u}{\partial x} + u + bu^2 - u^3\right]. \tag{1}$$

Together with (1) we study periodic edge conditions of

$$u(t, x + 2\pi) \equiv u(t, x). \tag{2}$$

In (1) we perform the substitution: $u(t, x) = \upsilon(t, x) + c$.

As a result we obtain the boundary-value problem

$$\frac{\partial \upsilon}{\partial t} = -\frac{\partial^2}{\partial x^2}\left[\alpha\frac{\partial^2 \upsilon}{\partial x^2} + \lambda\frac{\partial \upsilon}{\partial x} + \beta\upsilon + \gamma\upsilon^2 - \upsilon^3\right], \tag{3}$$

$$\upsilon(t, x + 2\pi) \equiv \upsilon(t, x), \tag{4}$$

where $\beta = 1 + 2bc - 3c^2,\ \gamma = b - 3c$. It is important to note that the condition

$$M(\upsilon(t_0, x)) \equiv \frac{1}{2\pi}\int_0^{2\pi} \upsilon(t_0, x)dx = 0$$

at all $t > t_0$ leads to the fact that the condition

$$M(\upsilon(t, x)) = 0 \tag{5}$$

holds.

While studying the local dynamics of the boundary-value problem (3)—(5) the important role is played by the arrangement of roots of Ak of the standard equation for the linearized in zero boundary-value problem:

$$\lambda_k = -\alpha k^4 + ik^3\lambda + \beta k^2, \qquad k = \pm 1, \pm 2, \dots \tag{6}$$

Below we suppose that a critical case takes place. Let the value of c = co be such that

$$\alpha = \beta = 1 + 2bc_0 - 3c_0^2. \tag{7}$$

Fix arbitrarily the value of c_1 and assume in (3)

$$c = c_0 + \varepsilon c_1, \tag{8}$$

where ε is a small positive parameter, i. e.

$$0 < \varepsilon \ll 1. \tag{9}$$

We study the behavior of all the solutions of the boundary-value problem (3)—(5) from some sufficiently small and not depending on e neighborhood of zero balance state under the conditions (7)—(9).

In this case the standard equation has a pair of pure imaginary roots of $\lambda_{\pm 1} = \pm i\lambda + O(\varepsilon)$, and all its remaining roots have negative (and separated from the imaginary axis) real parts. Thus, the conditions of Andronov–Hopf bifurcation are fulfilled.

Introduce into consideration a formal series

$$\upsilon = \varepsilon^{1/2}\big[\xi(\tau)\exp(ix + i\lambda t) + \bar{\xi}(\tau)\exp(-ix - i\lambda t)\big] + \varepsilon \upsilon_2(t,\tau,x) + \\ + \varepsilon^{3/2}\upsilon_3(t,\tau,x) + \dots \tag{10}$$

Here $\tau = \varepsilon t$ —is slow "time"; the functions $\upsilon_j(t,\tau,x)$ are $2\pi/\lambda$-periodic in t and 2π-periodic in x.

Substitute (10) into (3)—(5) and start equating the coefficients at like powers of ε. At the third step for finding $\upsilon_3(t,\tau,x)$ we come the boundary-value problem. The solvability condition of the boundary-value problem in the indicated class of functions is in performing the equality

$$\frac{d\xi}{d\tau} = \delta\xi + \sigma\xi|\xi|^2, \tag{11}$$

where

$$\delta = 2\gamma c_1, \quad \sigma = 2A\gamma - 3, \quad A = 2\gamma[8\alpha - 2\beta_0 + 3i\lambda]^{-1}.$$

From this and from the general statements we have the following result.

Theorem 1. *Let $\delta \neq 0$ and $Re\sigma \neq 0$. Then at all sufficiently small ε the dynamics of the equation* (11) *defines the local dynamics of the boundary-value problem* (3)—(5).

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MULTIPLICATIVE DYNAMICAL SYSTEMS IN TERMS OF THE INDUCED DYNAMICS

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An example of induced dynamics is realized by means of the new multiplicative determinant relation, roots of which give positions of particles. We give both: generic scheme of description of completely integrable dynamical system, parametrized by an arbitrary $N \times N$-matrix of momenta, and explicit model that interpolates between hyperbolic systems of Calogero–Moser and Ruijsenaars–Schneider. Some special cases of this model are detailed.

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DYNAMICS OF A CYLINDER AND TWO POINT VORTICES IN AN IDEAL FLUID

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The motion of point vortices are widely studied in the literature. We consider plane-parallel motion of a circular cylinder interacting dynamically with two point vortices. The system has 4 degrees of freedom and is governed by a set of ordinary differential equations which prove to be Hamiltonian. The governing equations are invariant under rigid transformations of the plane and thus admit three additional integrals of motion (besides the Hamiltonian function). On the zero level of linear momenta the three integrals are in involution which allows one to reduce the system's order by three units thereby obtaining a one-degree-of freedom Hamiltonian system. The latter is then investigated to reveal some intriguing features of the dynamics. The first author was supported by the Russian Foundation for Basic Research (project no. 20-01-00312 A). The second author was supported by the Russian Foundation for Basic Research (project no. 20-01-00399 A).

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ON ELECTRIC POTENTIAL OF THIN ROUND LAMELLA

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Let one consider the following two-dimensional problem on a disk with radius a:

$$\Delta\varphi = -\frac{I_0}{\lambda}\,\delta(\mathbf{r}-\mathbf{r}_0) + \frac{2}{\lambda}\sum_{k=1}^{N} I_k\,\delta(\mathbf{r}-\mathbf{r}_k), \tag{1}$$

where $\varphi(\mathbf{r})$ is electric potential on the thin disk, Δ is Laplacian, λ is surface conductivity of the disk, I_0 is current flowing into point $\mathbf{r}_0 = (b\,\cos\beta,\, b\,\sin\beta)$, $0 \leq b < a$, and I_k are currents flowing out via points $\mathbf{r}_k = (a\,\cos\theta_k,\, a\,\sin\theta_k)$, $k = \overline{1,\, N}$.

These constant currents obey to the following relation expressing conservation of charge in the system:

$$\sum_{k=0}^{N} I_k = I_0\,. \tag{2}$$

Equation (1) ought to be provided by boundary condition corresponding to impe-
netrability of disk's boundary Γ [1]:

$$\frac{\partial\varphi}{\partial n}|_\Gamma = 0\,, \tag{3}$$

where Γ is circle $|\mathbf{r}| = a$ with eliminated points $\mathbf{r}_k$, $k = \overline{1,\, N}$.

The next theorem has been proven:

Theorem. Exact solution of the problem (1)-(3) is equal to $\varphi(\mathbf{r}) = \varphi_s(\mathbf{r}) + \psi(\mathbf{r})$ where

$$\varphi_s(\mathbf{r}) = -\frac{I_0}{2\,\pi\,\lambda}\,\ln\frac{|\mathbf{r}-\mathbf{r}_0|}{a} + \sum_{k=1}^{N}\frac{I_k}{\pi\,\lambda}\,\ln\frac{|\mathbf{r}-\mathbf{r}_k|}{a}$$

is singular part of electric potential on the disk $|\mathbf{r}| < a$ and

$$\psi(\mathbf{r}) = -\frac{I_0}{4\,\pi\,\lambda}\,\ln\left[1 + \left(\frac{b\,r}{a^2}\right)^2 - 2\,\frac{b\,r}{a^2}\,\cos(\theta - \beta)\right]$$

is its regular part written in polar coordinates $(r,\,\theta)$.

Proof. Corner stones of the proof are equation (2) and a number of the well-known Fourier series.

The result obtained is very useful for technological control under measurement of surface conductivity λ by means of the Kelvin probe force microscopy (see [1] and references there in).

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ON THE LIMITATION OF BACKWARD INTEGRATION METHOD IN CASE OF RESONANCE ORBITS

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It is well known that backward integration method is restricted by few million years due to orbital chaoticity, (Laskar et al, 2011, Radovic,2017). But in case orbits close to resonance valid backward integration interval may be shorter.In particular,we note it in our previous works (Rosaev, 2019).

In this paper we have shown that results of backward and forward integration of the same orbit crossed the mean motion resonance in the presence of semimajor axis drift are not the same. We confirm the Murray Dermott (1999) theory conclusion that orbit migrated toward the planet can be (temporary) captured in resonance, when the same orbit diverged from the perturbed planet only jump from one side resonance to another. As a result, orbit before resonance crossing cannot be reconstructed properly by numeric integrations. This result significantly restricted on the applicability of the backward integration method to the interaction of the small asteroids with resonance in the presence of the Yarkovsky effect. Also, it can be important in the case of the studying of the planetary migrations.

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ON THE DIRICHLET PROBLEM FOR AN ELLIPTIC FUNCTIONAL-DIFFERENTIAL EQUATION WITH AFFINE TRANSFORMATION OF THE ARGUMENT

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The paper considers the Dirichlet problem for an elliptic functional-differential equation containing a combination of shifts and contractions of the argument of an unknown function under the Laplace operator. Sufficient conditions for unique solvability are established. It is also shown that the problem can have an infinite-dimensional variety of solutions.

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NONWANDERING SET POSSESSING WADA PROPERTY

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Let dissipative dynamic system ψ acting on Euclidean plane is constituted to be the diffeomorphism action at iterations. Moreover there subsists nonwandering continuum Υ being common boundary more then two regions. It is clear that $\psi_k(\Upsilon) = \Upsilon$ for all $k \in \mathbb{N}$.

The dynamic system is said to be *possesses Wada property* if there subsists invariant (nonwandering) continuum being common boundary three or more then regions.

There exist only two topological type, symmetric and antisymmetric, of dissipative dynamic system with nonwandering continuum being common boundary three region [1, 2]. It is clear that the dissipative dynamic system has either saddle and two unstable antisaddle fixed points or fixed inverse saddle and two two-periodic points.

Antisymmetric dynamic system with nonwandering continuum can transform to be dynamic system with nonwandering vortex street without fixed points. The further factorization procedure allows you to get a dynamic system possesses the Wada property with nonwanderig continuum being common boundary any finite regions number.

A special place among the obtained dynamic systems is occupied by a system with a nonwandering continuum being common boundary two regions and it being Birkhoff curve. It is positive solution of problem 1100 [3]: *does there exist an analytic diffeomorphism $f\colon A \to A$ without periodic points such that such that for some x in the interior of annulus A, the omega- limit of x contains points of both boundary components of annulus A?*

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NON ORDINARY GUTKIN BILLIARDS

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Consider ordinary Birkhoff billiard such that every trajectory leaving the curve with angle δ will next hit the boundary with angle δ again, regardless to the exit point. In the plane such examples were introduced by E. Gutkin in [2] and are very explicit. These type of billiards are related to the problem of floating bodies in equilibrium, which goes back to S. Ulam (problen 19 in [1]). We refer to these examples as ordinary Gutkin billiards and present non-ordinary Gutkin billiards. First result is an explicit example of a convex plane curve such that the outer billiard has a given finite number of invariant curves. In the second result we construct examples of magnetic Gutkin billiards where we consider magnetic billiards under a constant magnetic field.

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SYMMETRIES OF THE FULL SYMMETRIC TODA SYSTEM ON REAL LIE ALGEBRAS

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The full symmetric Toda system is the dynamical system determined by a Cartan involution on a real form of semisimple Lie algebra. It is known that the system is Hamiltonian and integrable. However, the construction of the first integrals of this system is not an easy task. In my talk, based on the joint work with Yu. Chernyakov and A. Sorin I will describe a way to construct a large commutative family of symmetries of this system, i.e. of vector fields, that will commute with each other and with the vector field that generates the system. The construction is based on the geometric considerations and on the structure of representations of the Lie algebra.

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RETROGRADE COORBITAL MOTION OF CELESTIAL BODIES: THE INVESTIGATION OF QUALITATIVE PROPERTIES USING WISDOM'S "ADIABATIC APPROXIMATION"

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In the last decades of the XXth century it became clear that behavior of approximate integrals of motion called adiabatic invariants is important in study of resonant effects in dynamics of celestial bodies (e.g. [1-3]). A first step of the standard scheme of the adiabatic approximation in the investigations of mean motion resonances (MMR) is averaging over the fastest dynamic process, i.e. over the orbital motion of the objects in commensurability. In averaged equations of motion one should take a subsystem that describes the process of "intermediate" time scale - the variation of the resonant angle. This subsystem can be interpreted as a one degree of freedom Hamiltonian system depending on other variables as slowly varying parameters. Consequently, the value of the "action" variable for this subsystem will be an adiabatic invariant (AI). Studying then the properties of level surfaces of AI in the subspace of the slowest variables we can draw conclusions about the secular evolution of orbits of celestial bodies in MMR.

As an illustration of the opportunities of this approach, we apply it to analyze the properties of retrograde co-orbital motion. Dynamically it corresponds to retrograde 1:1 MMR. This situation is not something abstract: the asteroid 514107 Ka'epaoka'awela is moving in such a resonance with the motion of Jupiter.

The author was supported by the Russian Foundation for Basic Research (project no. 20-01-00312A).

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CHAOTIC WAVE PACKET PROPAGATION IN DISORDERED NONLINEAR LATTICES WITH ONE AND TWO SPATIAL DIMENSIONS

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In this talk I will present some resent results on the chaotic behavior of several multidimensional, disordered, nonlinear Hamiltonian systems, emphasizing the quantification of chaos strength through the computation of the maximum Lyapunov exponent (mLE, see for example [1] and references therein). More specifically, I will discuss the dynamics of the disordered variants of two typical lattice models: the Klein-Gordon oscillator chain and the discrete nonlinear Schrödinger equation, focusing on the version of these systems for two spatial dimensions. Presenting results concerning the chaoticity of these models I will study the time evolution of the mLE and the distribution of the associated deviation vector [2, 3, 4]. I will emphasize the fact that the observed power law decays of the mLE have exponents different from -1, which is seen in the case of regular motion. The fact that the same dynamical behaviors are observed for both models signifies the generality of the underlying chaotic mechanisms.

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DISCRIMINANT SET OF THE RESTRICTED THREE-VORTEX PROBLEM ON A PLANE

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The dynamics of a few point vortices in a perfect fluid is a classical object for investigations starting from Kirchhoff, Thomson and Gröbli. We consider a restricted three-vortex problem on the unbounded plane when one of the vortices is fixed. The system has 2 degrees of freedom and is governed by a set of ordinary differential equations which can be present in a Hamiltonian form. The governing equations has an additional first integral. We extract a discriminant set of the system in an explicit form. Some critical motions of the vortices is also observed.

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NON-ABELIAN EVOLUTION SYSTEMS WITH CONSERVATION LAWS AND SYMMETRIES

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In this talk, we find noncommutative analogs for well-known polynomial evolution systems with higher conservation laws and symmetries. The integrability of obtained non-Abelian systems is justified by explicit zero curvature representations with spectral parameter.

The talk is based on joint papers with V. Adler.

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GRAPH COMBINATORICS, STATISTICAL PHYSICS AND CLUSTER ALGEBRAS

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The talk is focused on three related topics: electrical networks, polynomial graph invariants and the Ising-type models. All of them possesse the same characteristic properties to be an integrable statistical model, to be a cluster variety and to be inherent to some nontrivial combinatorial problem. I will explain the similarity between these tasks and present the substantial algebraic structure behind. The talk is based on some recent works with V. Gorbounov https://arxiv.org/abs/1905.03522 and B. Bychkov and A. Kazakov https://arxiv.org/abs/2005.10288.

ENTROPY OF AN OPERATOR

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We extend the concept of the measure entropy from the group of automorphisms of a measure space to the group of unitary operators on a Hilbert space. Our main motivations concern formalization of the idea of quantum chaos. The key ingredient of our construction is a (probably) new concept from functional analysis, the dimension of a (bounded) operator.

The author was supported by a grant from the Russian Science Foundation (project no. 18-01-00887).

RIMANN-ROCH THEOREM AND SUPERINTEGRABLE SYSTEMS

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The Riemann-Roch theorem for a divisor D on an irreducible genus g algebraic curve C is

$$dim|D| = deg(D) - g + i(D).$$

In classical mechanics positive divisors without matching points

$$D = P_1 + \ldots + P_n\,, \qquad deg(D) = n$$

appear in a study of dynamical systems integrable by Abel's quadratures

$$\sum_{i=1}^{n} \int^{P_i} \omega_j = t_j\,, \qquad j = 1, \ldots, n.$$

In this case

- $deg(D)$ is a number of variables of separation, i.e. number of degrees of freedom;

- g is a topological genus of algebraic curve C defined by separated relations.

It is natural to ask a question about the mechanical meaning of the remaining terms in the Riemann-Roch theorem.

If $n > g$ we have $i(D) = 0$ and dimension of linear space or dimension of the space of meromorphic functions is equal to

$$dim|D| = n - g > 0.$$

In this case divisor D of degree n is reduced to divisor $\rho(D) \in Jac(C)$ of degree g.

We want to discuss what mean $dim|D|$ and reduced divisor $\rho(D)$ on an example of superintegrable systems associated with hyperelliptic and non hyperelliptic algebraic curves.

INTEGRABLE SYSTEMS IN MULTIDIMENSIONS

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One of the main current topics in the field of integrable systems concerns the existence of nonlinear integrable evolution equations in more than two spatial dimensions. The fact that such equations exist was proven by A.S. Fokas [1], who derived equations of this type in 4+2 dimensions, i.e., four spatial dimensions and two time dimensions. The associated initial value problem for such equations, where the dependent variables are specified for all space variables at $t_1 = t_2 = 0$, can be solved by means of a nonlocal d-bar problem. The next step in this program is to formulate and solve nonlinear integrable systems in 3+1 dimensions (i.e., with three space variables and a single time variable) in agreement with physical reality. The method we employ is to first construct a system in 4+2 dimensions, with the aim to reduce this then to 3+1 dimensions.

In this talk we focus on the Davey-Stewartson system [2] and the 3-wave interaction equations [3]. Both these integrable systems have their origins in fluid dynamics where they describe the evolution and interaction, respectively, of wave packets on a fluid surface. We start from these equations in their usual form in 2+1 dimensions (two space variables x, y and one time variable t) and we bring them to 4+2 dimensions by complexifying each of these variables. We solve the initial value problem of these equations in 4+2 dimensions. Subsequently, for the Davey-Stewartson system in the linear limit we reduce this analysis to 3+1 dimensions to comply with the natural world. Finally, we discuss the construction of the 3+1 reduction of the full nonlinear problem, which is currently under investigation.

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SYMMETRIES AND INTEGRABILITY OF DIFFERENCE EQUATIONS

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The existence of hierarchies of higher order symmetries, aka generalised symmetries, serves as a definition of integrability. It applies to differential, differential-difference and difference equations. At the same time it can be easily argued that there is no simple and systematic method to compute generalised symmetries of difference equations. Some methods/approaches exist but are not always so helpful. And on top of that, there is a lack of relevant symbolic algebra software packages compared to the plethora of software for the symmetry analysis of differential equations.

In this talk I am going to present a systematic and algorithmic way to compute generalised symmetries of difference equations. This approach exploits the theory of integrability conditions, employs Laurent and Taylor formal series of pseudo-difference operators and formulates algebraically the determining equations.

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SPATIOTEMPORAL SOLITON BULLET DYNAMICS IN MULTIMODE OPTICAL FIBERS

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In recent years, nonlinear pulse propagation in multimode fibers (MMFs) is has experienced a dramatic resurgence of interest, because of their potential for optical communications and high-power lasers. However, from the fundamental viewpoint several aspects still remain to be fully understood. Here we experimentally and theoretically studied the dynamics of high-energy (up to reaching the fiber damage threshold) spatiotemporal solitons in MMFs with a graded-index (GRIN) core profile. Intra-pulse Raman scattering leads to the fission of the initial femtosecond pulse into different multimode solitons, which undergo Raman self-frequency shift (SSFS) [1-2]. In our experiments, we revealed the presence of a new nonlinear propagation regime in MMFs, where stable spatiotemporal solitons are created by the fission of the initial pulse. Remarkably, these solitons have different amplitudes and wavelengths, but nearly equal time duration [3].

Numerical simulations were conducted to reproduce the phenomenon of fission using an exact 3D+1 vector model, including higher-order dispersion, Kerr and Raman nonlinearities. We also included a phenomenological two-photon absorption (TPA) term, to model the presence of nonlinear losses.

The measured output spectrum shows that the Raman-induced SSFS tends to saturate for energies higher than 200 nJ. This is due to the presence of high nonlinear loss in the first few cm of MM fiber, owing to multi-photon absorption by fiber defects and doping [4]. Two distinct multimode soliton propagation regimes exist: in the first, only weak linear losses are present; in the second, the output energy remains clamped to a nearly constant value. Remarkably, in the nonlinear loss regime, nearly all of the transmitted energy is funneled into high-energy spatiotemporal soliton pulses with a bell-shaped, high-quality beam profile. These results are of significant interest for the development of new, high-power laser soliton sources in the mid-infrared domain of the spectrum.

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